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Research Article

Study the Influence of Nonlocal Boundary Condition on the Difference Eigenvalue Problem for Elliptic Partial Differential Equation

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Abstract. This paper presents a study of the difference eigenvalue problem for elliptic partial differential equations with a differential type multipoint nonlocal boundary conditions. We formulate the stability analysis technique which is based on the spectral structure of the transition matrix which has different types of eigenvalues. We begin by studying the one-dimensional problem and generalize the results to the two-dimensional problems by appropriate difference operators with nonlocal conditions.

Keywords. Elliptic partial differential equation; Nonlocal boundary condition; Eigenvalues and eigenvectors problem; Finite difference method

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1. Introduction

In many articles, numerical methods have been considered for the solution of *partial differential equations* (PDEs) involving *nonlocal boundary conditions* (NBCs). Some examples of these articles include nonlocal problems for elliptic equations [1–3,26], elliptic-parabolic [4], hyperbolic [5,27], or hyperbolic-parabolic [6].

Many applied phenomena have been modeled as mathematical equations with NBCs (see, for example, work in [9, 18]). Many thermoelasticity problems are formulated as nonlocal problems [17, 19, 22]. Two of the most recent new mathematical models in biotechnology are presented in [7, 15]. A separate class of such nonlocal models are *boundary value problem* (BVP) for elliptic equation with NBCs [12].

The eigenvalue problem is one of the important problems related to differential equations with NBCs. The investigation of the spectra for one and two-dimensional differential operators with NBCs of Bitsadze-Samarskii type, or integral-type are given in [8, 13, 14, 16, 23, 24]. Further results of PDEs with one Bitsadze-Samarskii NBC are published in [23, 25].

In this work, we consider the elliptic PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad \text{in } \Omega, \tag{1}$$

where $\Omega = (0,1)\times(0,1)$, with the boundary conditions

$$u(x,0) = u_1(x), \tag{2}$$

$$u(x,1) = u_2(x), \tag{3}$$

$$\tfrac{\partial u}{\partial x}(0,y) = \gamma_1 \tfrac{\partial u}{\partial x}(\varrho, y), \tag{4}$$

$$u(1,y) = \gamma_2 u(\zeta, y), \tag{5}$$

where ϱ, ζ, γ_1 and γ_2 are given constants such that $0 < \varrho < \zeta < 1$. We examine the difference eigenvalue problem corresponding to problem (1)-(5). We define the uniform grids ω_h and ω_k

$$\omega_h = \{x_i : x_i = ih,\ i = 0,1,\dots,N\},$$

$$\omega_k = \{y_j : y_j = jk,\ j = 0,1,\dots,M\},$$

where $h = \frac{1}{N}$, $k = \frac{1}{M}$, for positive integers N and M. Then, the grid $\omega_{h\times k}$ is defined by

$$\omega_{h\times k} = \omega_h \times \omega_k = \{(x_i, y_j) : x_i \in \omega_h,\ y_j \in \omega_k\}.$$

h is chosen such that ϱ and ζ are points on the grid ω_h, i.e. $\varrho = s_1 h$, $\zeta = s_2 h$, for positive integers s_1 and s_2. First, we study the following one dimensional difference eigenvalue problem with NBCs

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \tag{6}$$

$$u_0 = u_1 - \gamma_1 u_{s_1} + \gamma_1 u_{s_1-1}, \tag{7}$$

$$u_N = \gamma_2 u_{s_2}. \tag{8}$$

Hence, the one-dimensional relations obtained are employed to deduce similar results for the corresponding two-dimensional problem of the form

$$\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} + \lambda u_i^j = 0, \tag{9}$$

$$u_i^0 = 0, \tag{10}$$

$$u_i^N = 0, \tag{11}$$

$$u_0^j = u_1^j - \gamma_1 u_{s_1}^j + \gamma_1 u_{s_1-1}^j, \tag{12}$$

$$u_N^j = \gamma_2 u_{s_2}^j. \tag{13}$$

At certain values of λ, problem (6)-(8) or problem (9)-(13) will have a nontrivial solution. These values are called the eigenvalues of the problem. Since conditions (7)-(8) and (12)-(13) are nonlocal, the eigenvalues may assume real or complex values as the difference operator is non-self adjoint. The main idea of this study is to show the effect of the multipoint NBCs on different types of eigenvalues and (when it is possible) provide them with analytical expressions. We use techniques which are used, for example, in papers [10, 11, 20–22, 24] that explored similar problems with other types of NBCs.

The plan of this paper is as follows: Section 2, presents the structure of the matrix of the deference systems. Also, for one-dimensional we presented cases for the eigenvalues and their corresponding eigenvectors. In Section 3, the relations from Section 2 are generalized to the structure of the spectrum of two-dimensional differential and appropriate difference operators with nonlocal conditions. Section 4 lists the conclusion of this work.

2. The Difference Eigenvalue Problem in One-Dimension

Consider the case $M = N$. Then the linear system of equations (6)-(8) is defined by the following square matrix A of order $(N-1)\times(N-1)$ as

$$A = \frac{1}{h^2}\begin{pmatrix} 1 & -1 & 0 & 0 & -\gamma_1 & \gamma_1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_2 & \cdots & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\gamma_2 & \cdots & 0 & -1 \end{pmatrix},$$

where $-\gamma_1$ and γ_1 occupy the columns $s_1 - 1$ and s_1, respectively and $-\gamma_2$ occupies column s_2. Then, the finite difference eigenvalue problem (6)-(8) and the following eigenvalue problem

$$Au = \lambda u, \tag{14}$$

are equivalent. In matrix A, because of the NBCs some rows and columns are altered from the standard matrix of elliptic PDE with classical conditions. Hence, matrix A can have different types of eigenvalues according to the parameters h, ϱ, ζ, γ_1 and γ_2 and these cases are analyzed in the following work. We rewrite Equation (6) in the form

$$u_{i-1} - 2\left(1 - \frac{\lambda h^2}{2}\right)u_i + u_{i+1} = 0, \tag{15}$$

which is utilized in the following cases.

Lemma 2.1. *One of the following cases are provided the existence of the difference eigenvalue problem* (6)-(8) *has zero eigenvalues.*

(i) *if* $\gamma_1 = 1$ *and* $\gamma_2 \neq \frac{1}{\varrho}$, *then the difference eigenvector according to the difference eigenvalue is given by*

$$u_i = \left(-\frac{1-\gamma_2\zeta}{1-\gamma_2} + ih\right)c.$$

(ii) *if* $\gamma_1 = 1$ *and* $\gamma_2 = \frac{1}{\varrho}$, *then the difference eigenvector according to the difference eigenvalue is given by* $u_i = c(ih)$.

(iii) *if* $\gamma_1 \neq 1$ *and* $\gamma_2 = 1$. *In this case, then the difference eigenvector according to the difference eigenvalue is given by* $u_i = c$, *for an arbitrary constant* c.

Proof. If $\lambda = 0$, the solution of difference eigenvalue problem (6) is $u_i = c_1 + c_2 ih$, $i = 0, 1, 2, \cdots, N$. Then, substituting in (7) yields

$$(h - \gamma_1 h)c_2 = 0. \tag{16}$$

The second NBCs (8) yields,

$$(1 - \gamma_2)c_1 + (1 - \gamma_2(\zeta))c_2 = 0. \tag{17}$$

We have three cases: first if $\gamma_1 = 1$ and $\gamma_2 \neq \frac{1}{\varrho}$ which yields $c_2 \neq 0$ and $c_1 = -\frac{1-\gamma_2\zeta}{1-\gamma_2}c_2$ then, $u_i = \left(-\frac{1-\gamma_2\zeta}{1-\gamma_2} + ih\right)c$. But in case two $\gamma_1 = 1$, if $c_2 \neq 0$ and $c_1 = 0$ which yields $\gamma_2 = \frac{1}{\varrho}$ and $u_i = c(ih)$. Then in case three $\gamma_1 \neq 1$ and $\gamma_2 = 1$, if $c_2 = 0$ and $c_1 \neq 0$, than, $u_i = c$ where c an arbitrary constant. □

Lemma 2.2. *One of the two following cases are provided the existence of difference eigenvalue problem* (6)-(8), *which has a negative eigenvalue* $\lambda = -\frac{4}{h^2}\sinh^2\left(\frac{\alpha h}{2}\right)$, *where the positive parameter* α *satisfies the relation between* γ_1 *and* γ_2.

(i) *If* $\gamma_1 = \dfrac{1 - \cosh(\alpha h)}{\cosh(\alpha(\varrho - h)) - \cosh(\alpha\varrho)}$ *and* $\gamma_2 = 1$.

The difference eigenvector according to the difference eigenvalue is given by $u_i = c\cosh(\alpha ih)$.

(ii) *If* $\gamma_1 \neq \dfrac{1 - \cosh(\alpha h)}{\cosh(\alpha(\varrho - h)) - \cosh(\alpha\varrho)}$ *and*

$$\gamma_2 = \frac{1 - \cosh(\alpha h) + \sinh(\alpha h) - (-\cosh(\alpha\varrho) + \cosh((-h+\varrho)\alpha) + \sinh(\alpha\varrho) - \sinh((-h+\varrho)\alpha))\gamma_1}{\zeta - \zeta\cosh(\alpha h) + \sinh(\alpha h) + (\zeta\cosh(\varrho\alpha) - \zeta\cosh((-h+\varrho)\alpha) - \sinh(\varrho\alpha) + \sinh((-h+\varrho)\alpha))\gamma_1},$$

and the difference eigenvector according to the difference eigenvalue is given by

$$u_i = c\left(-\frac{\sinh(\alpha h) - \gamma_1\sinh(\alpha\varrho) + \gamma_1\sinh(\alpha(\varrho - h))}{\cosh(\alpha h) - \gamma_1\cosh(\alpha\varrho) + \gamma_1\cosh(\alpha(\varrho - h) - 1)}\cosh(\alpha hi) + \sinh(\alpha hi)\right).$$

Proof. If $\lambda < 0$, we have

$$1 - \frac{\lambda h^2}{2} > 1.$$

Denote

$$\cosh(\alpha h) = 1 - \frac{\lambda h^2}{2},$$

and putting this expression for the finite difference equation (15) into the form

$$u_{i-1} - 2\cosh(\alpha h)u_i + u_{i+1} = 0.$$

Then,

$$u_i = c_1\cosh(\alpha h i) + c_2\sinh(\alpha h i).$$

By substituting the above equation into NBCs (7) and (8), we obtain the two following equations in the unknowns c_1 and c_2.

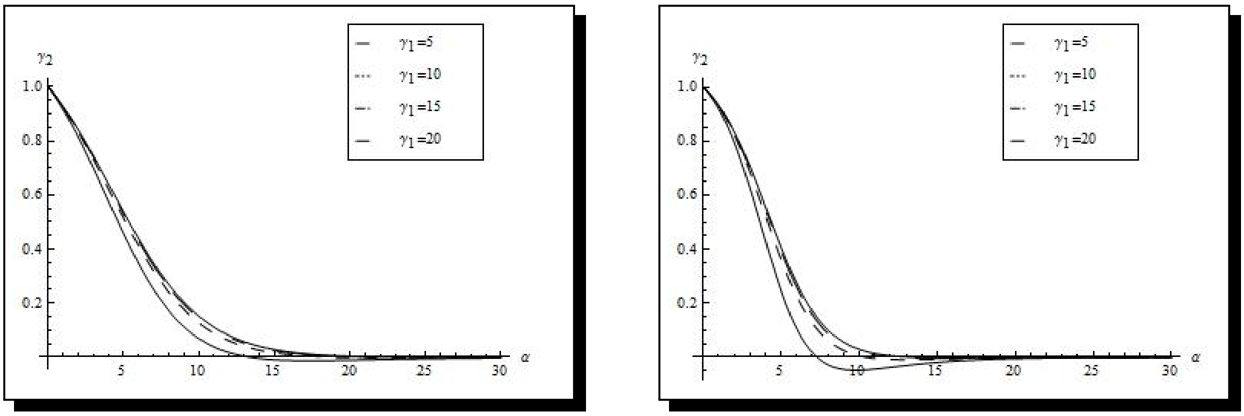
$$(\cosh(\alpha h)-\gamma_1\cosh(\alpha\varrho)+\gamma_1\cosh(\alpha(\varrho-h))-1)c_1 \\ +(\sinh(\alpha h)-\gamma_1\sinh(\alpha\varrho)+\gamma_1\sinh(\alpha(\varrho-h)))c_2=0, \tag{18}$$

$$(1-\gamma_2)c_1+(1-\gamma_2(\zeta))c_2=0, \tag{19}$$

when $\gamma_1=\dfrac{1-\cosh(\alpha h)}{\cosh(\alpha(\varrho-h))-\cosh(\alpha\varrho)}$ and $c_2=0$, which yields $\gamma_2=1$, and $u_i=c_1\cosh(\alpha i h)$.

But when $\gamma_1\neq\dfrac{1-\cosh(\alpha h)}{\cosh(\alpha(\varrho-h))-\cosh(\alpha\varrho)}$, then by solving the system of the two linear algebraic equations (18)-(19), we get

$$\gamma_2=\frac{1-\cosh(\alpha h)+\sinh(\alpha h)-(-\cosh(\alpha\varrho)+\cosh((-h+\varrho)\alpha)+\sinh(\alpha\varrho)-\sinh((-h+\varrho)\alpha))\gamma_1}{\zeta-\zeta\cosh(\alpha h)+\sinh(\alpha h)+(\zeta\cosh(\varrho\alpha)-\zeta\cosh((-h+\varrho)\alpha)-\sinh(\varrho\alpha)+\sinh((-h+\varrho)\alpha))\gamma_1}. \tag{20}$$



(a) $\varrho=0.2$, $\zeta=0.6$ **(b)** $\varrho=0.3$, $\zeta=0.8$

Figure 1. Effect of changing ϱ and ζ in equation (20) on the relation between γ_2 and α, (a) $\varrho=0.2$, $\zeta=0.6$, (b) $\varrho=0.3$, $\zeta=0.8$

Figure 1 shows that the different values of α will be close for value of γ_2.

Lemma 2.3. *One of the two following cases is provided the existence of difference eigenvalue problem* (6)-(8)*, has positive eigenvalues* $0<\lambda<\frac{4}{h^2}$*, by taking the form* $\lambda_k=\frac{4}{h^2}\sin^2\left(\frac{\alpha_k h}{2}\right)$*, where the parameters* $\alpha_k\in\left(0,\frac{\pi}{h}\right)$ *satisfy the relation between* γ_1 *and* γ_2.

(i) *If* $\gamma_1=\dfrac{1-\cos(\alpha h)}{\cos(\alpha(\varrho-h))-\cos(\alpha\varrho)}$ *and* $\gamma_2=\dfrac{\cos(\alpha)}{\cos(\zeta\alpha)}$.

The difference eigenvector according to the difference eigenvalue is given by $u_i=c\cos(\alpha i h)$.

(ii) *If* $\gamma_1\neq\dfrac{1-\cos(\alpha h)}{\cos(\alpha(\varrho-h))-\cos(\alpha\varrho)}$ *and*

$$\gamma_2=\frac{\sin((h-1)\alpha)-\sin(\alpha)-(-\sin((1-\varrho)\alpha)+\sin((h+1-\varrho)\alpha))\gamma_1}{\sin((h-\zeta)\alpha)+\sin(\zeta\alpha)+(\sin(\zeta-\varrho)\alpha)-\sin((h-\zeta-\varrho)\alpha))\gamma_1},$$

then the difference eigenvector according to the difference eigenvalue is given by

$$u_i=c(-\frac{\sin(\alpha h)-\gamma_1\sin(\alpha\varrho)+\gamma_1\sin(\alpha(\varrho-h))}{\cos(\alpha h)-\gamma_1\cos(\alpha\varrho)+\gamma_1\cos(\alpha(\varrho-h)-1)}\cos(\alpha h i)+\sin(\alpha h i)).$$

Proof. If $0 < \lambda < \frac{4}{h^2}$, then we have

$$\left|1 - \frac{\lambda h^2}{2}\right| < 1.$$

Denote

$$cos(\alpha h) = 1 - \frac{\lambda h^2}{2},$$

and putting this expression for the finite difference equation (15) into the form

$$u_{i-1} - 2\cos(\alpha h) u_i + u_{i+1} = 0.$$

Then,

$$u_i = c_1 \cos(\alpha h i) + c_2 \sin(\alpha h i).$$

By substituting the above equation into NBCs (7) and (8), we obtain the two following equations in the unknowns c_1 and c_2.

$$[\cos(\alpha h) - \gamma_1 \cos(\alpha \varrho) + \gamma_1 \cos(\alpha(\varrho - h)) - 1]c_1 + [\sin(\alpha h) - \gamma_1 \sin(\alpha \varrho) + \gamma_1 \sin(\alpha(\varrho - h))]c_2 = 0, \quad (21)$$

$$(\cos(\alpha) - \gamma_2 \cos(\alpha\zeta))c_1 + (\sin(\alpha) - \gamma_2 \sin(\alpha\zeta))c_2 = 0. \quad (22)$$

When $\gamma_1 = \dfrac{1-\cos(\alpha h)}{\cos(\alpha(\varrho - h)) - \cos(\alpha s_1 h)}$ and $c_2 = 0$, which yields: $\gamma_2 = \dfrac{\cos(\alpha)}{\cos(\zeta\alpha)}$, and $u_i = c\cos(\alpha i h)$.

But when $\gamma_1 \neq \dfrac{1-\cos(\alpha h)}{\cos(\alpha(\varrho - h)) - \cos(\alpha\varrho)}$, then by solving the system of the two linear algebraic equations (21)-(22), we get

$$\gamma_2 = \frac{\sin((h-1)\alpha) - \sin(\alpha) - (-\sin((1-\varrho)\alpha) + \sin((h+1-\varrho)\alpha))\gamma_1}{\sin((h-\zeta)\alpha) + \sin(\zeta\alpha) + (\sin(\zeta - \varrho)\alpha) - \sin((h-\zeta-\varrho)\alpha))\gamma_1}. \quad (23)$$

Lemma 2.4. *One of the two following cases is provided the existence of difference eigenvalue problem* (6)-(8)*, which has eigenvalue* $\lambda = \frac{4}{h^2}$.

(i) *If* $\gamma_1 = \dfrac{2}{(-1)^{s_1-1} - (-1)^{s_1}}$ *and* $\gamma_2 = \dfrac{(-1)^N}{(-1)^{s_2}}$.

The difference eigenvector according to the difference eigenvalue is given by $u_i = (-1)^i c$.

(ii) *If* $\gamma_1 \neq \dfrac{2}{(-1)^{s_1-1} - (-1)^{s_1}}$ *and*

$$\gamma_2 = \frac{-(-1)^{N+1}(1-\varrho) + (-1)^N(1) + \gamma_1((-1)^{N+s_1}(1-\varrho) - (-1)^{N+s_1-1}(1-\varrho+h) - (-1)^{s_2}(\zeta))}{\gamma_1((-1)^{s_2+s_1}(\zeta-\varrho) - (-1)^{s_2+s_1-1}(\zeta+\varrho-h)) - (-1)^{s_2+1}(\zeta-h)},$$

then the difference eigenvector according to the difference eigenvalue is given by

$$u_i = (-1)^i c\left(-\frac{(-1)h - \gamma_1(-1)^{s_1}(\varrho) + \gamma_1(-1)^{s_1-1}(\varrho-h)}{(-1) - \gamma_1(-1)^{s_1} + \gamma_1(-1)^{s_1-1} - 1} + ih\right).$$

Proof. If $\lambda = \frac{4}{h^2}$, we have in this lemma, the finite difference equation (15) has formed to

$$u_{i+1} + 2u_i + u_{i-1} = 0.$$

Then,

$$u_i = (-1)^i (c_1 + c_2(ih)),$$

and by substituting the above equation into NBCs (7) and (8), we obtain the two following equations in the unknowns c_1 and c_2 by take the form

$$((-1)-\gamma_1(-1)^{s_1}+\gamma_1(-1)^{s_1-1}-1)c_1+((-1)h-\gamma_1(-1)^{s_1}(\varrho)+\gamma_1(-1)^{s_1-1}(\varrho-h))c_2=0, \tag{24}$$

$$((-1)^N-(-1)^{s_2}\gamma_2)c_1+((-1)^N(1)-(-1)^{s_2}\gamma_2(\varrho))c_2=0. \tag{25}$$

When $\gamma_1=\dfrac{2}{(-1)^{s_1-1}-(-1)^{s_1}}$, then $c_2=0$. This yields $\gamma_2=\dfrac{(-1)^N}{(-1)^{s_2}}$, and $u_i=(-1)^i c_1$.

But when $\gamma_1\neq\dfrac{2}{(-1)^{s_1-1}-(-1)^{s_1}}$, then by solving the system of the two linear algebraic equations (24)-(25), we get

$$\gamma_2=\frac{-(-1)^{N+1}(1-\varrho)+(-1)^N(1)+\gamma_1((-1)^{N+s_1}(1-\varrho)-(-1)^{N+s_1-1}(1-\varrho+h)-(-1)^{s_2}(\zeta))}{\gamma_1((-1)^{s_2+s_1}(\zeta-\varrho)-(-1)^{s_2+s_1-1}(\zeta+\varrho-h))-(-1)^{s_2+1}(\zeta-h)}. \tag{26}$$

Lemma 2.5. *One of the two following cases is provided the existence of difference eigenvalue problem* (6)-(8)*, has positive eigenvalues $\lambda>\frac{4}{h^2}$ by taking the form $\lambda=\frac{4}{h^2}\cosh^2(\frac{\alpha h}{2})$, where the positive parameter α satisfies the relation between γ_1 and γ_2.*

(i) *If* $\gamma_1=\dfrac{1-(-1)\cosh(\alpha h)}{(-1)^{s_1-1}\cosh(\alpha(\varrho-h)-(-1)^{s_1}\cosh(\alpha\varrho)}$ *and* $\gamma_2=\dfrac{(-1)^N\cosh(\alpha)}{(-1)^{s_2}\cosh(\zeta\alpha)}$.

The difference eigenvector according to the difference eigenvalue is given by $u_i=(-1)^i c\cosh(\alpha ih)$.

(ii) *If* $\gamma_1\neq\dfrac{1-(-1)\cosh(\alpha h)}{(-1)^{s_1-1}\cosh(\alpha(\varrho-h)-(-1)^{s_1}\cosh(\alpha\varrho)}$ *and*

$$\gamma_2=\frac{-(-1)^N(\cosh(\varrho\alpha)\sinh(h\alpha)-(1+\cosh(h\alpha))\sinh(\alpha)-\frac{1}{2}(-1)^{N+s_1}\gamma_1}{2(-1)^{s_2}\cosh(\frac{h\alpha}{2})(\sinh(\frac{1}{2}(h-2\zeta)\alpha)}$$
$$\cdot\frac{(2\cosh((-h+\varrho)\alpha)-\sinh(2\varrho\alpha)+2\cosh(\varrho\alpha)(\sinh(\alpha)-\sinh((-h+\varrho)\alpha)))}{-(-1)^{s_1}\sinh(\frac{1}{2}(h+2\zeta-2\varrho)\alpha)\gamma_1)}.$$

Then the difference eigenvector according to the difference eigenvalue is given by

$$u_i=(-1)^i c(-\frac{(-1)\sinh(\alpha h)-\gamma_1(-1)^{s_1}\sinh(\alpha\varrho)+\gamma_1(-1)^{s_1-1}\sinh(\alpha(\varrho-h))}{(-1)\cosh(\alpha h)-\gamma_1(-1)^{s_1}\cosh(\alpha\varrho)+\gamma_1(-1)^{s_1-1}\cosh(\alpha(\varrho-h))-1}$$
$$\cdot\cosh(\alpha ih)+\sinh(\alpha ih)).$$

Proof. If $\lambda>\frac{4}{h^2}$. Let

$$1-\frac{\lambda h^2}{2}=-\cosh(\alpha h),$$

and by substituting this expression in (15) yields

$$u_{i-1}+2\cosh(\alpha h)u_i+u_{i+1}=0.$$

Then,

$$u_i=(-1)^i(c_1\cosh(\alpha h i)+c_2\sinh(\alpha h i)),$$

by substituting the above equation into NBCs (7) and (8), we obtain the two following equations in the unknowns c_1 and c_2.

$$((-1)\cosh(\alpha h)-\gamma_1(-1)^{s_1}\cosh(\alpha\varrho)+\gamma_1(-1)^{s_1-1}\cosh(\alpha(\varrho-1)-1))c_1$$

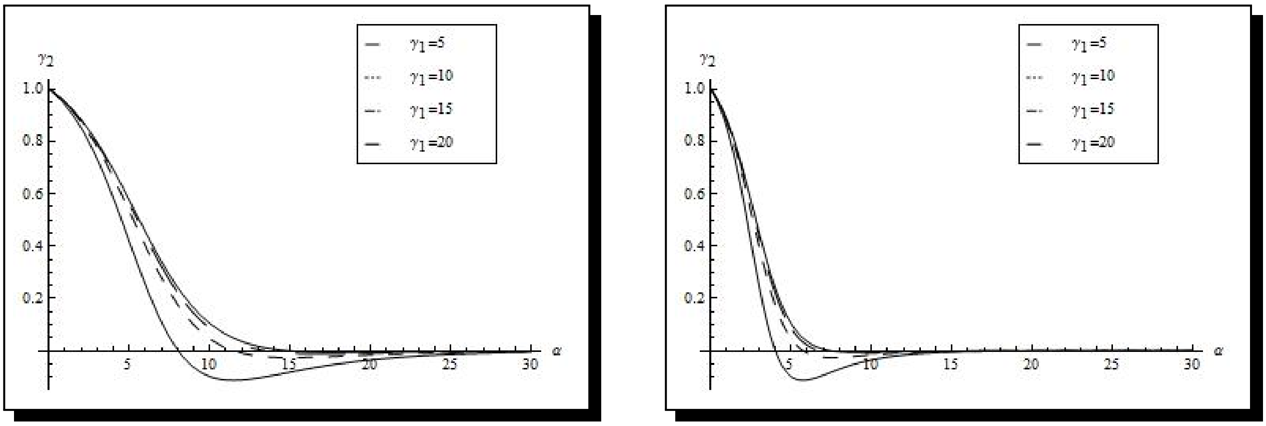
$$+((-1)\sinh(\alpha h)-\gamma_1(-1)^{s_1}\sinh(\alpha\varrho)+\gamma_1(-1)^{s_1-1}\sinh(\alpha(\varrho-h)))c_2=0, \tag{27}$$

$$((-1)^N\cosh(\alpha)-(-1)^{s_2}\gamma_2\cosh(\alpha\zeta))c_1+((-1)^N\sinh(\alpha)-(-1)^{s_2}\gamma_2\sinh(\alpha\zeta))c_2=0. \tag{28}$$

When $\gamma_1=\dfrac{1-(-1)\cosh(\alpha h)}{(-1)^{s_1-1}\cosh(\alpha(\varrho-h)-(-1)^{s_1}\cosh(\alpha\varrho)}$, then $c_2=0$. So, to obtain a nontrivial solution we get $\gamma_2=\dfrac{(-1)^N\cosh(\alpha)}{(-1)^{s_2}\cosh(\zeta\alpha)}$, and $u_i=(-1)^i c\cosh(\alpha i h)$.

But when $\gamma_1\neq\dfrac{1-(-1)\cosh(\alpha h)}{(-1)^{s_1-1}\cosh(\alpha(\varrho-h)-(-1)^{s_1}\cosh(\alpha\varrho)}$, then by solving the system of the two linear algebraic equations (27)-(28), then we get

$$\gamma_2=\frac{-(-1)^N(\cosh(\varrho\alpha)\sinh(h\alpha)-(1+\cosh(h\alpha))\sinh(\alpha)-\frac{1}{2}(-1)^{N+s_1}\gamma_1}{2(-1)^{s_2}\cosh(\frac{h\alpha}{2})(\sinh(\frac{1}{2}(h-2\zeta)\alpha)}$$
$$\cdot\frac{(2\cosh((-h+\varrho)\alpha)-\sinh(2\varrho\alpha)+2\cosh(\varrho\alpha)(\sinh(\alpha)-\sinh((-h+\varrho)\alpha)))}{-(-1)^{s_1}\sinh(\frac{1}{2}(h+2\zeta-2\varrho)\alpha)\gamma_1)}. \tag{29}$$



(a) $\varrho=0.2$, $\zeta=0.8$ **(b)** $\varrho=0.4$, $\zeta=0.8$

Figure 2. Effect of changing ϱ and ζ in equation (29) on the relation between γ_2 and α, (a) $\varrho=0.2$, $\zeta=0.8$, (b) $\varrho=0.4$, $\zeta=0.8$

Figure 2 shows that the different values of α will be close for value of γ_2. □

Lemma 2.6. *The complex eigenvalue of problem* (6)-(8) *takes the form* $\lambda_k=\frac{4}{h^2}\sin^2(\frac{q_k h}{2})$*, where* q_k *are the nontrivial complex numbers that satisfy*

$$\gamma_2=\frac{-e^{-hq-q-qh-q\varrho}(e^{q(h+\varrho)}(-e^{2q}-e^{qh}+e^{hq+qh})+e^{2q+qh})+e^{qh})(-1+e^{hq})(e^{(h+2)q}+e^{2q\varrho})\gamma_1}{e^{-hq-\zeta q-qh-q\varrho}(e^{q(h+\varrho)})(e^{2\zeta q}+e^{qh}-e^{hq+qh}+e^{2\zeta q+qh})+e^{qh}(-1+e^{hq})(e^{(h+2\zeta)q}+e^{2q\varrho})\gamma_1}.$$

The difference eigenvector according to the difference eigenvalue is given by

$$u_i=c\left(-\frac{e^{-qh}-\gamma_1 e^{-q\varrho}+\gamma_1 e^{-q(\varrho-h)}-1}{e^{qh}-\gamma_1 e^{q\varrho}+\gamma_1 e^{q(\varrho-h)}-1}e^{iqh}+e^{-iqh}\right),$$

where c *is an arbitrary constant.*

Proof. Denote the complex $\iota=\sqrt{-1}$ and consider $q=\alpha+\iota\beta$. Note that case $\alpha\neq0$, and $\beta\neq0$. As the other two cases synchronize with Lemma 2.2 and Lemma 2.3. Also, when $\alpha=0$ and $\beta=0$, the status is same as in Lemma 2.1.

Let $\cos(qh) = 1 - \frac{\lambda h^2}{2}$. Then by substituting this expression in (15) yields

$$u_{i-1} - 2\cosh(qh)u_i + u_{i+1} = 0.$$

Then,

$$u_i = c_1 e^{qhi} + c_2 e^{-(qhi)}.$$

By substituting the above equation into NBCs (7) and (8), we obtain the two following equations in the unknowns c_1 and c_2.

$$(e^{qh} - \gamma_1 e^{q\varrho} + \gamma_1 e^{q(\varrho-h)} - 1)c_1 + (e^{-qh} - \gamma_1 e^{-q\varrho} + \gamma_1 e^{-q(\varrho-h)} - 1)c_2 = 0. \tag{30}$$

$$(e^{q} - \gamma_2 e^{q\zeta})c_1 + (e^{-q} + \gamma_2 e^{-q\zeta})c_2 = 0. \tag{31}$$

The situation of the γ_2 equation produced from a nontrivial solution when the determination of this system is equal to zero. When solving the two equations (30) and (31), we get α and β values. □

3. The Eigenvalues and Eigenvectors of the Two-Dimension Problem

Consider the difference eigenvalue problem (9)-(13). By using separation of variables technique, we assume

$$u_{ij} = v_i z_j, \quad i,j = 0,1,2,\cdots,N,$$

we obtain two one-dimensional eigenvalue problems

$$\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} + \mu_i v_i = 0, \quad v_{s_1} = 0,\ v_{s_2} = \gamma v_{N-s_2}, \tag{32}$$

and

$$\frac{z_{j-1} - 2z_j + z_{j+1}}{h^2} + \omega j z_j = 0, \quad z_0 = 0,\ z_N = 0, \tag{33}$$

where $\lambda = \lambda_{k,\ell} = \mu_k + \omega_\ell$. The eigenvalues of (33) are real, positive and are given by the formula [10].

$$\omega_\ell = \frac{4}{h^2}\sin^2\left(\frac{\ell\pi h}{2}\right), \quad \ell = 1,2,\cdots,N-1, \tag{34}$$

and the difference eigenvector according to the difference eigenvalue is given by

$$z_\ell = (z_\ell)_j = \sin\left(\frac{\ell\pi h j}{2}\right), \quad \ell = 1,2,\cdots,N-1.$$

Corollary 3.1. *If the problem* (9)-(13) *has positive eigenvalues* $0 < \lambda_{k,\ell} < \frac{8}{h^2}$, *if they exist at all, then they are given by the formula*

$$\lambda_{k,\ell} = \frac{4}{h^2}\left(\sin^2\left(\frac{\alpha_k h}{2}\right) + \sin^2\left(\frac{\ell\pi h}{2}\right)\right), \quad k = 1,2,\cdots,N-1.$$

Then, the corresponding eigenvectors are occurred in two cases:

(i) *If* $\gamma_1 = \dfrac{1-\cos(\alpha h)}{\cos(\alpha(\varrho-h)) - \cos(\alpha\varrho)}$ *and* $\gamma_2 = \dfrac{\cos(\alpha)}{\cos(\zeta\alpha)}$, *are given by*

$$(u_{k,\ell})_{i,j} = c\cos(\alpha_k ih)\sin\left(\frac{\ell\pi hj}{2}\right),$$

whereas in the case

(ii) $\gamma_1 \neq \dfrac{1-\cos(\alpha h)}{\cos(\alpha(\varrho-h))-\cos(\alpha\varrho)}$ *and*

$$\gamma_2 = \frac{\sin((h-1)\alpha)-\sin(\alpha)-(-\sin((1-\varrho)\alpha)+\sin((h+1-\varrho)\alpha))\gamma_1}{\sin((h-\zeta)\alpha)+\sin(\zeta\alpha)+(\sin((\zeta-\varrho)\alpha)-\sin((h-\zeta-\varrho)\alpha))\gamma_1},$$

are given by

$$(u_{k,\ell})_{i,j} = c\left(-\frac{\sin(\alpha_k h)-\gamma_1\sin(\alpha_k\varrho)+\gamma_1\sin(\alpha_k(\varrho-h))}{\cos(\alpha_k h)-\gamma_1\cos(\alpha_k\varrho)+\gamma_1\cos(\alpha_k(\varrho-h)-1)}\cos(\alpha_k hi)+\sin(\alpha_k hi)\right)\sin\left(\frac{\ell\pi hj}{2}\right),$$

where α_k are the roots of equation (23) *and all indices are form* 1 *to* $N-1$.

Corollary 3.2. *If the problem* (9)-(13) *has positive eigenvalues, if they exist at all, then they have the form*

$$\widetilde{\lambda_\ell} = \frac{4}{h^2}\left(1+\sin^2\left(\frac{\pi\ell h}{2}\right)\right), \quad \ell = 1,2,\cdots,N-1,$$

and the corresponding eigenvectors are occurred in two cases:

(i) *If* $\gamma_1 = \dfrac{2}{(-1)^{s-1}-(-1)^{s_1}}$ *and* $\gamma_2 = \dfrac{(-1)^N}{(-1)^{s_2}}$, *are given by*

$$(u_\ell)_{i,j} = (-1)^i c\sin\left(\frac{\ell\pi hj}{2}\right),$$

whereas in the case

(ii) $\gamma_1 \neq \dfrac{2}{(-1)^{s_1-1}-(-1)^{s_1}}$ *and*

$$\gamma_2 = \frac{-(-1)^{N+1}(1-\varrho)+(-1)^N(1)+\gamma_1((-1)^{N+s_1}(1-\varrho)-(-1)^{N+s_1-1}(1-\varrho+h)-(-1)^{s_2}(\zeta))}{\gamma_1((-1)^{s_2+s_1}(\zeta-\varrho)-(-1)^{s_2+s_1-1}(\zeta+\varrho-h))-(-1)^{s_2+1}(\zeta-h)},$$

and given by

$$(u_\ell)_{i,j} = (-1)^i c\left(-\frac{(-1)h-\gamma_1(-1)^{s_1}(\varrho)+\gamma_1(-1)^{s_1-1}(\varrho-h)}{(-1)-\gamma_1(-1)^{s_1}+\gamma_1(-1)^{s_1-1}-1}+ih\right)\sin\left(\frac{\ell\pi hj}{2}\right),$$

where all the indices are from 1 *to* $N-1$.

Corollary 3.3. *If the problem* (9)-(13) *has positive eigenvalues, if they exist at all, then they can be formed by*

$$\overline{\lambda_\ell} = \frac{4}{h^2}\left(\cosh^2\left(\frac{\alpha h}{2}\right)+\sin^2\left(\frac{\pi\ell h}{2}\right)\right), \quad \ell = 1,2,\cdots,N-1,$$

and the corresponding eigenvectors are occurred in two cases:

(i) *If* $\gamma_1 = \dfrac{1-(-1)\cosh(\alpha h)}{(-1)^{s_1-1}\cosh(\alpha(\varrho-h)-(-1)^{s_1}\cosh(\alpha\varrho)}$ *and* $\gamma_2 = \dfrac{(-1)^N\cosh(\alpha)}{(-1)^{s_2}\cosh(\zeta\alpha)}$, *are given by*

$$(u_\ell)_{i,j} = (-1)^i c\cosh(\alpha ih)\sin\left(\frac{\ell\pi hj}{2}\right),$$

whereas in the case

(ii) $\gamma_1 \neq \dfrac{1-(-1)\cosh(\alpha h)}{(-1)^{s_1-1}\cosh(\alpha(\varrho-h)-(-1)^{s_1}\cosh(\alpha\varrho)}$ *and*

$$\gamma_2 = \frac{-(-1)^N(\cosh(\varrho\alpha)\sinh(h\alpha)-(1+\cosh(h\alpha))\sinh(\alpha)+\frac{1}{2}(-1)^{N+s_1}\gamma_1}{2(-1)^{s_2}\cosh(\frac{h\alpha}{2})(\sinh(\frac{1}{2}(h-2\zeta)\alpha)}$$

$$\cdot \frac{(2\cosh((-h+\varrho)\alpha)-\sinh(2\varrho\alpha)+2\cosh(\varrho\alpha)(\sinh(\alpha)-\sinh((-h+\varrho)\alpha)))}{-(-1)^{s_1}\sinh(\frac{1}{2}(h+2\zeta-2\varrho)\alpha)\gamma_1)},$$

and the eigenvector is given by

$$(u_\ell)_{i,j} = (-1)^i c\Big(-\frac{(-1)\sinh(\alpha h)-\gamma_1(-1)^{s_1}\sinh(\alpha\varrho)+\gamma_1(-1)^{s_1-1}\sinh(\alpha(\varrho-h))}{(-1)\cosh(\alpha h)-\gamma_1(-1)^{s_1}\cosh(\alpha\varrho)+\gamma_1(-1)^{s_1-1}\cosh(\alpha(\varrho-h))-1}$$
$$\cdot\cosh(\alpha ih)+\sinh(\alpha ih)\Big)\sin\left(\frac{\ell\pi hj}{2}\right),$$

where α are the positive roots of the equation (29) *and all the indices are from* 1 *to* $N-1$.

We note that for problem (32), the unique negative eigenvalue takes the form

$$\mu = -\frac{4}{h^2}\sinh^2\left(\frac{\alpha h}{2}\right),$$

where α is the positive root of equation (20). Then, as

$$\alpha_\ell^* = \frac{2}{h}\log\left(\sin\left(\frac{\pi\ell h}{2}\right)+\sqrt{\sin^2\left(\frac{\pi\ell h}{2}\right)+1}\right), \quad \ell = 1,2,\cdots,N-1,$$

are the positive roots of the equations

$$\sinh^2\left(\frac{\alpha h}{2}\right) = \sin^2\left(\frac{\alpha h}{2}\right), \quad \ell = 1,2,\cdots,N-1,$$

the following statement is valid.

Corollary 3.4. *If $\lambda_{k,\ell}=0$, then one of the two following cases provided that the problem* (9)-(13) *has an algebraically simple zero eigenvalue*

(i) *If $\gamma_1 = \dfrac{1-\cosh(\alpha h)}{\cosh(\alpha(\varrho-h))-\cosh(\alpha\varrho)}$ and $\gamma_2 = 1$, the difference eigenvector according to the difference eigenvalue is given by*

$$(u_\ell)_{i,j} = c\cosh(\alpha^* ih)\sin\left(\frac{\ell\pi hj}{2}\right).$$

(ii) *If $\gamma_1 \neq \dfrac{1-\cosh(\alpha h)}{\cosh(\alpha(\varrho-h))-\cosh(\alpha\varrho)}$ and*

$$\gamma_2 = \frac{1-\cosh(\alpha h)+\sinh(\alpha h)-(-1*\cosh(\alpha\varrho)+\cosh((-h+\varrho)\alpha)+\sinh(\alpha\varrho)-\sinh((-h+\varrho)\alpha))\gamma_1}{\zeta-\zeta\cosh(\alpha h)+\sinh(\alpha h)+(\zeta\cosh(\varrho\alpha)-\zeta\cosh((-h+\varrho)\alpha)-\sinh(\varrho\alpha)+\sinh((-h+\varrho)\alpha))\gamma_1},$$

and the difference eigenvector according to the difference eigenvalue is given by

$$(u_\ell)_{i,j} = c\Big(-\frac{\sinh(\alpha^* h)-\gamma_1\sinh(\alpha^*\varrho)+\gamma_1\sinh(\alpha^*(\varrho-h))}{\cosh(\alpha^* h)-\gamma_1\cosh(\alpha^*\varrho)+\gamma_1\cosh(\alpha^*(\varrho-h)-1)}$$
$$\cdot\cosh(\alpha^* hi)+\sinh(\alpha^* hi)\Big)\sin\left(\frac{\ell\pi hj}{2}\right).$$

If either one of the two conditions (i) *or* (ii) *is satisfied with α_r^* for a positive integer r, $1\le r\le N-1$, then problem* (9)-(13) *has $r-1$ negative eigenvalues*

$$\lambda_{r,\ell} = -\frac{4}{h^2}\left(\sinh^2\left(\frac{\alpha^* h}{2}\right)-\sin^2\left(\frac{\pi\ell h}{2}\right)\right), \quad \ell = 1,2,\cdots,s-1,$$

and an algebraically simple eigenvalue $\lambda_{r,r}=0$.

4. Conclusion

In this article, the stability analysis of difference schemes for a (one-dimensional and two-dimensional) elliptic partial differential equations with nonlocal boundary conditions. The two nonlocal boundary conditions are based on the spectral structure of the transition matrix of a difference scheme. The qualitative behavior of eigenvalues depending on the multipoint nonlocal conditions which effected on the different values of the eigenvalue, hence the corresponding eigenvectors. The relations can be combined from one-dimensional problems to obtain the corresponding ones of the two-dimensional by using the separation of variables technique.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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