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Research Article

On a Convergence Theorem for the General Noor Iteration Process in Uniformly Smooth Banach Spaces

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Abstract. In this paper, two different classes of mappings namely, uniformly continuous asymptotically nonexpansive and uniformly continuous asymptotically demicontractive mappings are considered on the general modified Noor iteration process with errors and proved to converge strongly to the fixed point of uniformly continuous asymptotically demicontractive mappings in uniformly smooth Banach spaces. The new result can be viewed as an improvement to a multitude of results in the fixed point theory especially those of Xu and Noor [8], Owojori and Imoru [5] and also the results of Owojori [6].

Keywords. Asymptotically nonexpansive mapping; Uniformly continuous asymptotically demicontractive mapping and Ishikawa iteration process with errors

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1. Introduction

Let K be a nonempty convex subset of a uniformly smooth Banach space E and let T be a selfmap of K. The set $F_T = \{x \in K : Tx = x\}$ is called the fixed point set of T in K.

A mapping $T:K\to K$ is called asymptotically nonexpansive if there exists a real sequence $\{k_n\}_{n=0}^{\infty}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n\|x-y\|, \quad \text{for all } x,y\in K,\ n\ge 1, \tag{1.1}$$

$T:K\to K$ is called asymptotically demicontractive if $F_T\neq\phi$ and there exists a sequence $\{k_n\}_{n=0}^{\infty}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and a constant k in $[0,1)$ such that

$$\|T^n x - p\|^2 \le k_n^2\|x-p\|^2 + k\|x-T^n x\|^2, \quad \text{for all } x\in K,\ p\in F_T,\ n\ge 1 \tag{1.2}$$

A Banach space $(E,\|\cdot\|)$ is called uniformly convex if, given $\epsilon>0$, there exists $\delta>0$ such that for all $x,y\in E$ with $\|x\|\le 1$, $\|y\|\le 1$ and $\|x-y\|\ge\epsilon$, we have $\left\|\frac{1}{2}(x+y)\right\|<1-\delta$.

E is called smooth if, for every $x\in E$ with $\|x\|=1$, there exists a unique f in its dual E^* such that $\|f\|=\langle f,x\rangle = 1$.

The modulus of smoothness of E is the function $\rho_E:[0,\infty)\to[0,\infty)$, defined by

$$\rho_E(\tau)=\sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : x,y\in E, \|x\|\le 1, \|y\|\le\tau\right\}.$$

E is called uniformly smooth if

$$\lim_{\tau\to\infty}\frac{\rho_E(\tau)}{\tau}=0$$

and, for $q>1$, E is said to be q-uniformly smooth if there exits a constant $c>0$ such that

$$\rho_E(\tau)\le c\tau^q, \quad \tau\in[0,\infty).$$

Several researchers have introduced and proved results on the general modified three-step iteration scheme for solving nonlinear equation

$$Tx = x$$

for asymptotically nonexpansive operators in Banach spaces.

Undoubtedly, one of such results established by Xu and Noor [8] is the strong convergence theorem of the modified scheme to the fixed point of asymptotically nonexpansive mappings in Banach spaces.

The result is as follows:

Theorem 1.1. *Let B be a uniformly convex Banach space and K a nonempty closed, bounded and convex subset of B.*
Let T be a completely continuous asymptotically nonexpansive selfmap of K with $\{k_n\}$ satisfying $k_n\ge 1$ and $\sum_{n=0}^{\infty} k_n<\infty$.
Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real numbers in $[0,1]$ satisfying

(i) $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$.

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

For a given $x_0\in D$, define sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n+\alpha_n T^n y_n\\ y_n &= (1-\beta_n)x_n+\beta_n T^n z_n\\ z_n &= (1-\gamma_n)x_n+\gamma_n T^n x_n \end{aligned}\right\} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$. Then, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ each converges strongly to a fixed point of T.

Remark 1.1. The proof of Theorem 1.1 is contained in Xu and Noor [8].

In 2003, Owojori and Imoru [5] introduced a modified three-step iteration scheme given by the following:

Theorem 1.2. *Let B be a uniformly convex Banach space and K a nonempty closed, bounded and convex subset of B. Suppose T is a uniformly continuous asymptotically nonexpansive selfmap of K. Let $\{k_n\}$ be a real sequence with $k_n \geq 1$ satisfying $\lim\limits_{n\to\infty} k_n = 1$ such that $k_n^p + 1 \leq p$, $p > 1$.*
For a given $x_1 \in K$, define sequence $\{x_n\}$ generated iteratively by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n T^n x_n \\ y_n &= a'_n x_n + b'_n T^n z_n + c'_n u_n \\ z_n &= a''_n x_n + b''_n T^n x_n + c''_n v_n \end{aligned}\right\} \tag{1.4}$$

for all $n \geq 1$, where $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are arbitrary sequences in K and $\{a_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{a''_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{b''_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{c''_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$;

(ii) $\sum\limits_{n=0}^{\infty} b_n = \infty$;

(iii) $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$, $\gamma_n = b''_n + c''_n$ *and* $\alpha_n[1 + \beta_n k_n^p(1 + \gamma_n k_n^p)] \leq \frac{1}{p-1-k_n^p}$.

Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Remark 1.2. The proof of Theorem 1.2 is contained in Owojori and Imoru [5].

Remark 1.3. A special case of the iteration scheme (1.4) is given by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T^n z_n + c'_n v_n \\ z_n &= a''_n x_n + b''_n T^n x_n + c''_n w_n \end{aligned}\right\} \tag{1.5}$$

for all $n \geq 1$, where $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ are bounded sequences in K and $\{a_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{a''_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{b''_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{c''_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$;

(ii) $\sum\limits_{n=0}^{\infty} b_n = \infty$.

In 2005, by using this special iteration scheme (1.6), Owojori [6] proved the following result:

Theorem 1.3. *Let X be an arbitrary real normed linear space and K a nonempty closed bounded and convex subset of X. Suppose $T : K \to K$ is a completely continuous asymptotically* *demicontractive mapping with real sequence* $\{k_n\}_{n=1}^{\infty} \subset [0,\infty)$ *satisfying* $k_n \geq 1$, *for all* n, $\lim_{n\to\infty} k_n = 0$ *and* $k_n^p + 1 < p$. *For a given* $x_1 \in K$, *define sequence* $\{k_n\}_{n=1}^{\infty}$ *generated iteratively by*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T^n z_n + c'_n v_n \\ z_n &= a''_n x_n + b''_n T^n x_n + c''_n w_n \end{aligned}\right\} \tag{1.6}$$

for all $n \geq 1$, *where* $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ *are arbitrary sequences in* K *and* $\{a_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{a''_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{b''_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{c''_n\}_{n=1}^{\infty}$ *are real sequences in* $[0,1]$ *satisfying the following conditions*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$;

(ii) $\sum_{n=0}^{\infty} b_n = \infty$, $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$, $\gamma_n = b''_n + c''_n$;

(iii) $k_n^2(1-\beta_n) + \beta_n k_n^4(1-\gamma_n) + \beta_n\gamma_n k^2 \leq 1$.

Then, the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Remark 1.4. The proof of Theorem 1.3 is contained in Owojori [6].

Our purpose in this paper is to use a more general modified Noor type iteration scheme with errors and establish the strong convergence of the three-step iteration scheme to the fixed point of asymptotically demicontractive mappings in uniformly smooth Banach spaces.

Two different classes of mappings namely, uniformly continuous asymptotically nonexpansive and uniformly continuous asymptotically demicontractive mappings are considered on the general modified Noor iteration process with errors and proved that the scheme converges strongly to the fixed point of uniformly continuous asymptotically demicontractive mappings in uniformly smooth Banach spaces.

2. Preliminaries

We shall employ the following Lemmas in the proof of our main result:

Lemma 2.1 ([9]). *Let* B *be a uniformly smooth Banach space. Then,* B *has modulus of smoothness of power type* $q \geq 1$, *if and only if there exists a constant* $c > 0$ *such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q, \quad \text{for all } x,y \in B. \tag{2.1}$$

Lemma 2.2 ([1,2,7]). *Let* B *be a uniformly smooth Banach space with modulus of smoothness of power type* $q \geq 1$. *Then, for all* x,y,z *in* B *and* $\lambda \in [0,1]$, *we have*

$$\|\lambda x + (1-\lambda)y - z\|^q \leq [1-\lambda(q-1)]\|y-z\|^q + \lambda c\|x-z\|^q - \lambda[1-\lambda^{q-1}c]\|x-y\|^q, \tag{2.2}$$

where c *is the constant appearing in* (2.1).

Lemma 2.3 ([3]). *Let* $\{\rho_n\}_{n=0}^{\infty}$ *be a nonnegative sequence of real numbers satisfying*

$$\rho_{n+1} \leq (1-\delta_n)\rho_n + \sigma_n, \tag{2.3}$$

where $\delta_n \in [0,1]$, $\sum \delta_n = \infty$ *and* $\sigma_n = o(\delta_n)$. *Then,* $\lim_{n\to\infty} \rho_n = 0$.

Lemma 2.4 ([4]). *Every asymptotically nonexpansive selfmapping of a uniformly smooth and convex Banach space has a fixed point.*

Lemma 2.5. *Let K be a closed convex subset of a Banach space E and $S : K \to K$ a continuous operator on K. Then, there exists a point $x^* \in K$ such that $Sx^* = x^*$.*

Remark 2.1. Hilbert spaces are uniformly smooth Banach spaces with modulus of smoothness of power type $q = 2$ and $c = 1$. For example, see Chidume and Osilike [1].

Remark 2.2. For x, y, z in a Hilbert space H with $q = 2$ and $c = 1$, inequality (2.2) gives

$$\|\lambda x + (1-\lambda)y - z\|^2 \le (1-\lambda)\|y - z\|^2 + \lambda\|x - z\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

that is,

$$\|(1-\lambda)(y - z) - \lambda(z - x)\|^2 \le (1-\lambda)\|y - z\|^2 + \lambda\|x - z\|^2 - \lambda(1-\lambda)\|x - y\|^2 . \tag{2.4}$$

The following is the main result in this paper:

3. Main Result

Theorem 3.1. *Let K be a nonempty closed, bounded and convex subset of a uniformly smooth Banach space E. Suppose S, T are selfmaps of K with S a uniformly continuous asymptotically nonexpansive mapping on K and T a uniformly continuous asymptotically demicontractive operator on K. For a given $x_1 \in K$, define a sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n S^n x_n \\ y_n &= a'_n x_n + b'_n S^n z_n + c'_n u_n \\ z_n &= a''_n x_n + b''_n T^n x_n + c''_n v_n \end{aligned}\right\} \tag{3.1}$$

for all $n \ge 1$, where $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are arbitrary sequences in K and $\{a_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{a''_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{b''_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{c''_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$;

(ii) $\sum_{n=0}^{\infty} b_n = \infty$;

(iii) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$;

(iv) $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$, $\gamma_n = b''_n + c''_n$;

(v) $\alpha_n(1 - k_n^2)[1 + \beta_n k_n^2(1 + \gamma_n k_n^2)] \le 1$.

Then, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Proof. Since T is asymptotically demicontractive, then $F_T \neq \phi$, that is, T has a fixed point in K. Denote it by $p \in K$.
Hence, by using (3.1), conditions (3.1)(i) and (iv) together with Remark 2.2, we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|a_n x_n + b_n T^n y_n + c_n S^n x_n - p\|^2 \\ &= \|(1-\alpha_n)x_n + (\alpha_n - c_n)T^n y_n + c_n S^n x_n - p\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|(1-\alpha_n)x_n - (1-\alpha_n)p + \alpha_n(T^n y_n - p) - c_n(T^n y_n - S^n x_n)\|^2 \\
&= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p) - c_n(T^n y_n - S^n x_n)\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n y_n\|^2 \\
&\quad + c_n\|T^n y_n - S^n x_n\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n y_n\|^2 \\
&\quad + c_n\|(T^n y_n - x_n) - (S^n x_n - x_n)\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n y_n\|^2 \\
&\quad + c_n\|T^n y_n - x_n\|^2 + c_n\|S^n x_n - x_n\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 + [c_n - \alpha_n(1-\alpha_n)]\|x_n - T^n y_n\|^2 \\
&\quad + c_n\|S^n x_n - x_n\|^2. \qquad (3.2)
\end{aligned}
$$

But, T is asymptotically demicontractive, therefore

$$\|T^n y_n - p\|^2 \le k_n^2\|y_n - p\|^2 + k\|y_n - T^n y_n\|^2. \qquad (3.3)$$

Substituting (3.3) into (3.2) yields

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n k_n^2\|y_n - p\|^2 + \alpha_n k\|y_n - T^n y_n\|^2 \\
&\quad + [c_n - \alpha_n(1-\alpha_n)]\|x_n - T^n y_n\|^2 + c_n\|x_n - S^n x_n\|^2. \qquad (3.4)
\end{aligned}
$$

Since K is bounded and T, S uniformly continuous on K, therefore, the sequences $\{\|x_n - T^n y_n\|\}$ and $\{\|x_n - S^n x_n\|\}$ are all bounded by some constant $M_1 < +\infty$.

Hence, (3.4) becomes

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n k_n^2\|y_n - p\|^2 + \alpha_n k\|y_n - T^n y_n\|^2 \\
&\quad + [2c_n - \alpha_n(1-\alpha_n)]M_1. \qquad (3.5)
\end{aligned}
$$

We now estimate $\|y_n - p\|^2$ in (3.5) as follows:

By using (3.1), conditions (3.1)(i), (iv) and Remark 2.2, to obtain

$$
\begin{aligned}
\|y_n - p\|^2 &= \|a'_n x_n + b'_n S^n z_n + c'_n u_n - p\|^2 \\
&= \|(1-\beta_n)x_n + (\beta_n - c'_n)S^n z_n + c'_n u_n - p\|^2 \\
&= \|(1-\beta_n)x_n - (1-\beta_n)p + \beta_n(S^n z_n - p) - c'_n(S^n z_n - u_n)\|^2 \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(S^n z_n - p) - c'_n(S^n z_n - u_n)\|^2 \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|S^n z_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - S^n z_n\|^2 \\
&\quad + c'_n\|S^n z_n - u_n\|^2. \qquad (3.6)
\end{aligned}
$$

Since K is bounded and S continuous on K, therefore, the sequences $\{\|x_n - S^n z_n\|\}$ and $\{\|S^n z_n - u_n\|\}$ are all bounded by some constant $M_2 < +\infty$.

Therefore, (3.6) gives

$$\|y_n - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|S^n z_n - p\|^2 + [c'_n - \beta_n(1-\beta_n)]M_2 \qquad (3.7)$$

K is a convex, closed and bounded subset of a uniformly smooth Banach space E and S a uniformly continuous asymptotically nonexpansive mapping on K, then by Lemma 2.4 (Goebel and Kirk [4]), we obtain that the set F_S of fixed points of S in K is nonempty, i.e. $F_S \ne \phi$. Since

$F_T \neq \phi$ and $F_S \neq \phi$, it follows that $F_T \cap F_S \neq \phi$. Therefore, let p be the common fixed point of S and T.
Then, (3.7) becomes

$$\|y_n - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|S^n z_n - S^n p\|^2 + [c'_n - \beta_n(1-\beta_n)]M_2. \tag{3.8}$$

But, S is asymptotically nonexpansive on K, therefore

$$\|S^n z_n - S^n p\|^2 \le k_n^2\|z_n - p\|^2. \tag{3.9}$$

Substitute (3.9) into (3.8) yields

$$\|y_n - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2\|z_n - p\|^2 + [c'_n - \beta_n(1-\beta_n)]M_2. \tag{3.10}$$

Next, we estimate $\|z_n - p\|^2$ in (3.10) as follows:
By (3.1), conditions (3.1)(i) and (iv) together with Remark 2.2, we get

$$\begin{aligned}
\|z_n - p\|^2 &= \|a''_n x_n + b''_n T^n x_n + c''_n v_n - p\|^2 \\
&= \|(1-\gamma_n)x_n + (\gamma_n - c''_n)T^n x_n + c''_n v_n - p\|^2 \\
&= \|(1-\gamma_n)x_n - (1-\gamma_n)p + \gamma_n(T^n x_n - p) - c''_n(T^n x_n - v_n)\|^2 \\
&= \|(1-\gamma_n)(x_n - p) + \gamma_n(T^n x_n - p) - c''_n(T^n x_n - v_n)\|^2 \\
&\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|T^n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2 \\
&\quad + c''_n\|T^n x_n - v_n\|^2.
\end{aligned} \tag{3.11}$$

T is asymptotically demicontractive, therefore

$$\|T^n x_n - p\|^2 \le k_n^2\|x_n - p\|^2 + k\|x_n - T^n x_n\|^2. \tag{3.12}$$

Substituting (3.12) into (3.11) gives

$$\begin{aligned}
\|z_n - p\|^2 &\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n k_n^2\|x_n - p\|^2 + \gamma_n k\|x_n - T^n x_n\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2 + c''_n\|T^n x_n - v_n\|^2 \\
&= (1 + \gamma_n k_n^2 - \gamma_n)\|x_n - p\|^2 + [\gamma_n k - \gamma_n(1-\gamma_n)]\|x_n - T^n x_n\|^2 \\
&\quad + c''_n\|T^n x_n - v_n\|^2.
\end{aligned} \tag{3.13}$$

But, K is bounded and T continuous on K, therefore, the sequences $\{\|x_n - T^n x_n\|\}$ and $\{\|T^n x_n - v_n\|\}$ are all bounded by some constant $M_3 < +\infty$.

Thus, (3.13) yields

$$\|z_n - p\|^2 \le (1 + \gamma_n k_n^2 - \gamma_n)\|x_n - p\|^2 + [c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3. \tag{3.14}$$

Substitute (3.14) into (3.10), to obtain

$$\begin{aligned}
\|y_n - p\|^2 &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2(1 - \gamma_n + \gamma_n k_n^2)\|x_n - p\|^2 \\
&\quad + \beta_n k_n^2[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3 + [c'_n - \beta_n(1-\beta_n)]M_2 \\
&= [(1-\beta_n) + \beta_n k_n^2(1 - \gamma_n + \gamma_n k_n^2)]\|x_n - p\|^2 + [c'_n - \beta_n(1-\beta_n)]M_2 \\
&\quad + \beta_n k_n^2[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3.
\end{aligned} \tag{3.15}$$

By using (3.1), conditions (3.1)(i) and (iv); and Remark 2.2, the estimate of $\|y_n - T^n y_n\|$ becomes

$$\|y_n - T^n y_n\|^2 = \|a'_n x_n + b'_n S^n z_n + c'_n u_n - T^n y_n\|^2$$

$$
\begin{aligned}
&= \|(1-\beta_n)x_n + (\beta_n - c'_n)S^n z_n + c'_n u_n - T^n y_n\|^2 \\
&= \|(1-\beta_n)x_n - (1-\beta_n)T^n y_n + \beta_n(S^n z_n - T^n y_n) - c'_n(S^n z_n - u_n)\|^2 \\
&= \left\|(1-\beta_n)(x_n - T^n y_n) + \beta_n(S^n z_n - T^n y_n) - c'_n(S^n z_n - u_n)\right\|^2 \\
&\le (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|S^n z_n - T^n y_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n z_n\|^2 + c'_n\|S^n z_n - u_n\|^2 \\
&= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n[\|(S^n z_n - x_n) - (T^n y_n - x_n)\|^2] \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n z_n\|^2 + c'_n\|S^n z_n - u_n\|^2 \\
&\le (1-\beta_n)\left\|x_n - T^n y_n\right\|^2 + \beta_n\|S^n z_n - x_n\|^2 + \beta_n\|T^n y_n - x_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n z_n\|^2 + c'_n\|S^n z_n - u_n\|^2 \\
&= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|x_n - S^n z_n\|^2 + \beta_n\|x_n - T^n y_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - S^n z_n\|^2 + c'_n\|S^n z_n - u_n\|^2 \\
&= \|x_n - T^n y_n\|^2 + \beta_n^2\|x_n - S^n z_n\|^2 + c'_n\|S^n z_n - u_n\|^2
\end{aligned}
\tag{3.16}
$$

Since K is bounded and T,S continuous on K, then the sequences $\{\|x_n - T^n y_n\|\}$, $\{\|x_n - S^n z_n\|\}$ and $\{\|S^n z_n - u_n\|\}$ are all bounded by some constant $M_4 < +\infty$.

Hence, (3.16) gives

$$\|y_n - T^n y_n\|^2 \le (1 + \beta_n^2 + c'_n)M_4. \tag{3.17}$$

Substitute (3.15) and (3.17) into (3.5), so that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n k_n^2[(1-\beta_n) + \beta_n k_n^2(1-\gamma_n+\gamma_n k_n^2)]\|x_n - p\|^2 \\
&\quad + \alpha_n k_n^2[c'_n - \beta_n(1-\beta_n)]M_2 + \alpha_n k_n^2(\beta_n k_n^2)[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3 \\
&\quad + \alpha_n k(1+\beta_n^2 + c'_n)M_4 + [2c_n - \alpha_n(1-\alpha_n)]M_1 \\
&= [(1-\alpha_n) + \alpha_n k_n^2(1-\beta_n) + \beta_n k_n^2(1-\gamma_n+\gamma_n k_n^2)]\|x_n - p\|^2 \\
&\quad + [2c_n - \alpha_n(1-\alpha_n)]M_1 + \alpha_n k_n^2[c'_n - \beta_n(1-\beta_n)]M_2 \\
&\quad + \alpha_n k_n^2(\beta_n k_n^2)[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3 + \alpha_n k(1+\beta_n^2 + c'_n)M_4 \\
&= \{1 - \alpha_n[(1-k_n^2)(1+\beta_n k_n^2(1+\gamma_n k_n^2))]\}\|x_n - p\|^2 \\
&\quad + [2c_n - \alpha_n(1-\alpha_n)]M_1 + \alpha_n k_n^2[c'_n - \beta_n(1-\beta_n)]M_2 \\
&\quad + \alpha_n k_n^2(\beta_n k_n^2)[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3 + \alpha_n k(1+\beta_n^2 + c'_n)M_4 \\
&= (1-\delta_n)\|x_n - p\|^2 + [2c_n - \alpha_n(1-\alpha_n)]M_1 + \alpha_n k_n^2[c'_n - \beta_n(1-\beta_n)]M_2 \\
&\quad + \alpha_n k_n^2(\beta_n k_n^2)[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3 + \alpha_n k(1+\beta_n^2 + c'_n)M_4,
\end{aligned}
\tag{3.18}
$$

where

$$\delta_n = \alpha_n[(1-k_n^2)(1+\beta_n k_n^2(1+\gamma_n k_n^2))]$$

and by exploiting condition (3.1)(v), and

$$\delta_n = \alpha_n[(1-k_n^2)(1+\beta_n k_n^2(1+\gamma_n k_n^2))] \le \alpha_n \le 1.$$

By our hypothesis, it is clear that $\delta_n \ge 0$. Thus, $0 \le \delta_n \le 1$. Also, by conditions (3.1)(ii) and (iv), we have $\sum \alpha_n = \infty$, which implies that $\sum \delta_n = \infty$.

Hence, (3.18) yields

$$\|x_{n+1}-p\|^2 \le (1-\delta_n)\|x_n-p\|^2 + [2c_n - \alpha_n(1-\alpha_n)]M_1 + \alpha_n k_n^2[c'_n - \beta_n(1-\beta_n)]M_2 + \alpha_n k_n^2(\beta_n k_n^2)[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)]M_3 + \alpha_n k(1+\beta_n^2 + c'_n)M_4. \tag{3.19}$$

Let $M_5 = \max\{M_1, M_2, M_3, M_4\}$. Then, (3.19) reduces to

$$\|x_{n+1}-p\|^2 \le (1-\delta_n)\|x_n-p\|^2 + \{[2c_n - \alpha_n(1-\alpha_n)] + \alpha_n k_n^2[c'_n - \beta_n(1-\beta_n)] + \alpha_n k_n^2(\beta_n k_n^2)[c''_n + \gamma_n k - \gamma_n(1-\gamma_n)] + \alpha_n k(1+\beta_n^2 + c'_n)\}M_5. \tag{3.20}$$

By using hypothesis (3.1)(iv) and by observing that $c_n < \alpha_n$, $c'_n < \beta_n$, $c''_n < \gamma_n$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\delta_n)\|x_n-p\|^2 + \{[2\alpha_n - \alpha_n(1-\alpha_n)] + \alpha_n k_n^2[\beta_n - \beta_n(1-\beta_n)] \\ &\quad + \alpha_n k_n^2(\beta_n k_n^2)[\gamma_n + \gamma_n k - \gamma_n(1-\gamma_n)] + \alpha_n k(1+\beta_n^2+\beta_n)\}M_5 \\ &= (1-\delta_n)\|x_n-p\|^2 + [\alpha_n + \alpha_n^2 + \alpha_n\beta_n^2 k_n^2 + \alpha_n\beta_n k_n^4(\gamma_n k + \gamma_n^2) + \alpha_n k(1+\beta_n+\beta_n^2)]M_5 \\ &= (1-\delta_n)\|x_n-p\|^2 + \alpha_n[1+\alpha_n+\beta_n^2 k_n^2 + \beta_n\gamma_n k_n^4(k+\gamma_n) + k(1+\beta_n+\beta_n^2)]M_5 \end{aligned} \tag{3.21}$$

is obtained.

Let $\sigma_n = \alpha_n[1+\alpha_n+\beta_n^2 k_n^2 + \beta_n\gamma_n k_n^4(k+\gamma_n) + k(1+\beta_n+\beta_n^2)]M_5$. Clearly, $\sigma_n = o(\delta_n)$.

In (3.21), put

$$\|x_n-p\|^2 = \rho_n,$$

then,

$$\rho_{n+1} \le (1-\delta_n)\rho_n + \sigma_n.$$

Hence, by Lemma 2.3 (Weng [3]), so that

$$\lim_{n\to\infty} \rho_n = 0$$

which implies that sequence $\{x_n\}$ converges strongly to p.

This completes the proof. □

Remark 3.1. Iteration (3.1) is more general than iterations (1.3) and (1.4) above in the following sense: By specializing some of the sequences and the selfmap in iteration (3.1) above, to obtain a general modified three-step iteration scheme of Xu and Noor [8].

Indeed, if $c_n = c'_n = c''_n = 0$ and $S = T$, then (3.1) will reduce to

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n \end{aligned}\right\}$$

where $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$ are real sequences in $[0,1]$ satisfying

$$a_n + b_n = a'_n + b'_n = a''_n + b''_n = 1$$

which is the same as equation (1.3) — the modified three-step iteration scheme of Xu and Noor [8], with $\alpha_n = b_n$, $\beta_n = b'_n$ and $\gamma_n = b''_n$.

Suppose the set S is defined by relation $S = T$ in equation (3.1), then (3.1) will become

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n T^n x_n \\ y_n &= a'_n x_n + b'_n T^n z_n + c'_n u_n \\ z_n &= a''_n x_n + b''_n T^n x_n + c''_n v_n \end{aligned}\right\}$$

for all $n \geq 1$, where $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are arbitrary sequences in K and $\{a_n\}_{n=1}^{\infty}$, $\{a'_n\}_{n=1}^{\infty}$, $\{a''_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, $\{b''_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{c'_n\}_{n=1}^{\infty}$, $\{c''_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying the condition:

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$$

which is the same as equation (1.4) — the modified three-step iteration scheme of Owojori and Imoru [5], with $\alpha_n = b_n + c_n$, $\beta_n = b'_n + c'_n$, $\gamma_n = b''_n + c''_n$ and $\alpha_n(1-k_n^2)(1+\beta_n\gamma_n k_n^2) \leq 1$, $p = 2 > 1$.

Therefore, the obtained result is clearly an improvement of that of Owojori and Imoru [5] as well as that of Xu and Noor [8]. Clearly, the result in Theorem 1.2 is a corollary to the new main result.

4. Conclusion

The class of uniformly continuous asymptotically nonexpansive mappings together with the class of uniformly continuous asymptotically demicontractive mappings are considered on the general modified Noor iteration process with errors and proved to converge strongly to the fixed point of uniformly continuous asymptotically demicontractive mappings in uniformly smooth Banach spaces.

Apart from these two classes of mappings, further research can be studied on other mappings in a general Banach space setting.

Further results can also be obtained by using the general modified Ishikawa iteration process with errors as consequences of the main result in this paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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