Research Article

Domination in Double Vertex Graphs

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Abstract. In this paper many bounds for the domination number of double vertex graph and its complement are obtained. Further we have obtained some characterizations of domination number of double vertex graph.

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1. Introduction

The game of chess was the main source behind the beginning of domination theory in graphs. The books by Haynes *et al.* [5] on domination in graphs encouraged the readers to take up the research work on domination theory. The domination theory in graphs has evolved in last few decades. Communication network, traffic network, organizational network requires a set of junctions which can control the rest of the network junctions. Hence it is enough we find a smaller set of junctions of a network that controls the rest. This can be achieved by using the concept of domination. Domination of cartesian product of graphs and some of the derived graphs are discussed in [8], [6] and [7]. As double vertex graph is an induced subgraph of Cartesian product of a graph with itself, the domination in double vertex graph is an interesting question to study.

In this paper $G = (V, E)$ is a finite, connected and simple graph of n vertices and m edges. For undefined terminologies and notations we refer [3]. Alavi *et al*. [2] introduced a graph

function which generates a new graph from a given graph G called double vertex graph of a graph G denoted as $U_2(G)$. The vertex set of $U_2(G)$ consists of unordered pairs from V such that two vertices $\{a,b\}$ and $\{c,d\}$ are adjacent in $U_2(G)$ if and only if $|\{a,b\} \cap \{c,d\}| = 1$ and if $a = c$ then b and d are adjacent in G. If $V(G) = \{v_i : 1 \le i \le n\}$, we describe vertex set of $U_2(G)$ as $V(U_2(G)) = \bigcup_{i=1}^{n-1} U_i$, where $U_i = \{\{v_i, v_j\}/v_i, v_j \in V(G); i+1 \le j \le n\}$. Double vertex graphs are investigated in [2], [4] and [1].
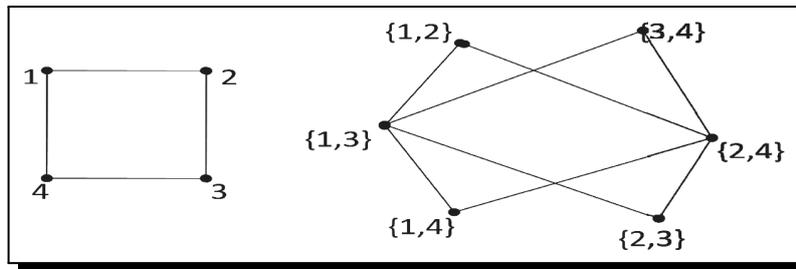

Figure 1

A subset D of V is called as a dominating set of G if every vertex not in D is adjacent to atleast one vertex in D. Further if no proper subset of D is dominating in G then D is minimal. The cardinality of a minimum dominating set of G is the domination number of G denoted by $\gamma(G)$. In general, the degree of a vertex w in a graph G is the number of edges incident with w in G, denoted as $\deg w$. A vertex of degree 1 is pendant vertex and a vertex adjacent to pendant vertex is support vertex. A vertex of degree $n-1$ is a full degree vertex. The maximum degree among the vertices of G is denoted by $\Delta(G)$ or Δ. For an edge $e = uw$ in G, the degree of the edge e is $\deg e = \deg u + \deg w - 2$ and maximum edge degree is denoted as $\Delta'(G)$. For a vertex v in G, the eccentricity ecc(v) of v is the distance between v and a vertex farthest from v in G. The maximum eccentricity among the vertices of G is its diameter denoted by diam(G).

Graph products can be used as a tool to systematically produce new graphs from a given graph or a set of graphs. The study of a graph product includes studying the properties of larger graph by studying those properties on a set of smaller graphs which are the factors of the larger graph. Here we investigate many bounds for the domination number of double vertex graph of G in terms of order, size, degree, edge degree and diameter of G. Further some bounds for the domination number of complement of double vertex graph are also obtained.

A vertex $\{a,b\}$ is said to be a *line pair* of $U_2(G)$ if and only if $ab \in E(G)$ otherwise a *non line pair* [4] and there by $V(U_2(G))$ can be divided into two sets U and W where U corresponds line pairs and W corresponds to non line pairs of $U_2(G)$ respectively.

For any vertex $\{h,k\}$ of $U_2(G)$,

$$\deg\{h,k\} = \begin{cases} \deg h + \deg k - 2 & \{h,k\} \in U \\ \deg h + \deg k & \{h,k\} \in W \end{cases}$$

Clearly, the degree of a vertex of $U_2(G)$ is atmost $2n-4$. Further if G is of order $n \ge 4$, $U_2(G)$ has atleast two non adjacent vertices. Therefore in this case $U_2(G)$ has no full degree vertex and $\gamma(U_2(G)) \ge 2$. However if $G \cong K_3, P_3$ then $U_2(G) \cong G$ and $\gamma(U_2(G)) = 1$.

2. Main Results

Theorem 2.1. *For a graph G, $\lfloor n/2 \rfloor \le \gamma(U_2(G))$.*

Proof. Since the degree of a vertex in $U_2(G)$ is atmost $2n-4$, any vertex of $U_2(G)$ can dominate atmost $2n-4$ vertices. A vertex $\{v_i, v_{i+1}\}$ of $U_2(G)$ can dominate vertices from sets U_i and U_{i+1} of $V(U_2(G))$. Define $D = \{\{v_i, v_{i+1}\} : i = 1, 3, 5, \ldots, n-1(n-2 \text{ for odd } n)\}$ which dominates $\bigcup_{i=1,3,5,\ldots}(U_i \cup U_{i+1}) = V(U_2(G)) - D$. Since no two vertices of D are adjacent no proper subset D' of D can dominate $V(U_2(G)) - D'$. Therefore, if $D(|D| = \lfloor n/2 \rfloor)$ is a possible dominating set then it is minimal. Hence $\lfloor n/2 \rfloor \le \gamma(U_2(G))$.
Equality is attained for $G \cong K_n, C_4, K_n - e, P_4, K_{1,3}$. □

Theorem 2.2. *For a graph G, $\gamma(G) + \gamma(U_2(G)) < \left(\frac{n+1}{2}\right)^2$.*

Proof. The bound for the domination number of G is given by

$$\gamma(G) \le \frac{n}{2}. \tag{2.1}$$

Therefore

$$\gamma(U_2(G)) \le \frac{\binom{n}{2}}{2}, \tag{2.2}$$

$$\Rightarrow \quad \gamma(G) + \gamma(U_2(G)) \le \frac{n}{2} + \frac{\binom{n}{2}}{2} \le \frac{n}{2} + \frac{n(n-1)}{4} < \left(\frac{n+1}{2}\right)^2. \tag{2.3}$$

□

Theorem 2.3. *For a graph G, if $\operatorname{diam}(G) \le 2$ then the set of line pairs U is dominating in $U_2(G)$.*

Proof. *Case* 1: Suppose $\operatorname{diam}(G) = 1$.
In this case $G \cong K_n$ and every vertex of $U_2(K_n)$ is a line pair. Then any vertex $\{v_i, v_{i+1}\}$ of D dominates $U_i \cup U_{i+1}$. Therefore, D dominates $\bigcup_{i=1,3,5,\ldots}(U_i \cup U_{i+1}) = V(U_2(G)) - D$. Now by Theorem 2.1 the set D is minimal.

Case 2: $\operatorname{diam}(G) = 2$.
For a non line pair $\{x, y\}$ in $U_2(G)$ there exists a vertex t in G such that the vertices x, t, y induces $P_3 : xty$ in G where line pairs $\{x, t\}$ and $\{t, y\}$ dominate the non line pair $\{x, y\}$. So U dominates W. Further if G has no triangle then U is an independent subset of $U_2(G)$. Consequently, U has no proper subset which is dominating. Thus in this case U is minimal. □

Theorem 2.4. *For a graph G, if $\operatorname{diam}(G) \ge 3$ then the set of line pairs U is not dominating in $U_2(G)$.*

Proof. Let $P_{k+1} : v_1 - v_2 - v_3 - \cdots - v_{k+1}$ be a path of length $k \ge 3$ in G. Then the vertex $\{v_1, v_{k+1}\}$ corresponding to the end vertices of P_{k+1} is a non line pair in $U_2(G)$ which is not adjacent to any of the line pairs. Therefore U is not dominating in $U_2(G)$. □

Theorem 2.5. *For a graph G, if $\operatorname{diam}(G) \ge 2$ and has no C_3 then the set of non line pairs W is dominating in $U_2(G)$.*

Proof. Let $\{x, y\}$ be a line pair of $U_2(G)$. Since $\text{diam}(G)$ is atleast two, there exists a vertex t in G such that t, x, y induces a path $P_3 : txy$ in G. Then $\{t, y\}$ is a non line pair that dominates the line pair $\{x, y\}$. Therefore, W dominates U. $\qquad\square$

Corollary 2.6. *For a graph G, if $\text{diam}(G) \geq 2$ and has no C_3 then $\gamma(U_2(G)) \leq \binom{n}{2} - m$. Equality is attained for $G \cong C_4$.*

Theorem 2.7. *For a graph G, if $\text{diam}(G) \geq 2$ and has no C_3 then $\lceil \frac{m}{2} \rceil \leq \gamma(U_2(G))$.*

Proof. By Theorem 2.5, W dominates U. Let R be the subset of W consisting of non line pairs which are end vertices of P_3 and S is the subset of $W - R$. Then $D = R \cup S$ dominates in $V(U_2(G)) - (R \cup S)$. Then $R \subseteq D$ which implies $\lceil \frac{m}{2} \rceil \leq \gamma(U_2(G))$.
Equality is attained for $G \cong C_4, P_4, C_5, K_{1,3}, K_{1,5}$. $\qquad\square$

Corollary 2.8. *For a tree, $\lceil \frac{n-1}{2} \rceil \leq \gamma(U_2(T)) \leq \binom{n-1}{2}$.*

Theorem 2.9. *For a graph, $\gamma(U_2(G)) \leq \binom{n}{2} - \Delta'(G)$.*

Proof. A line pair $\{x, y\}$ dominates $\Delta'(G)$ vertices in $U_2(G)$ where $\Delta'(G)$ is the maximum edge degree in G. Then for a dominating set D containing $\{x, y\}$, $\gamma(U_2(G)) \leq \binom{n}{2} - \Delta'(G)$. $\qquad\square$

Theorem 2.10. *For a graph G, $\gamma(U_2(G)) \leq \binom{n}{2} - \left(1 + \binom{\Delta}{2}\right)$ where Δ is the maximum degree of G.*

Proof. Let u be a vertex of G adjacent to the vertices $v_1, v_2, v_3, \ldots, v_\Delta$ and $\deg u = \Delta(G)$. Partition $V(U_2(G))$ into two disjoint sets X and Y where $X = \{\{u, v_i\} : 1 \leq i \leq \Delta\} \cup \{\{v_j, v_k\} : 1 \leq j \leq \Delta - 1; j + 1 \leq k \leq \Delta\}$, $Y = V(U_2(G)) - X$ and $|X| = \binom{\Delta+1}{2}$; $|Y| = \binom{n}{2} - \binom{\Delta+1}{2}$. The set X induces a subgraph isomorphic to $U_2(K_{1,\Delta})$ in $U_2(G)$. Let Z be a proper subset of X defined as $Z = \{\{u, v_j\} : 1 \leq j \leq \Delta - 2\} \cup \{\{v_\Delta, v_{\Delta-1}\}\}$ containing $\Delta - 1$ vertices dominates in X. Now for any minimum dominating set D of $U_2(G)$, $D \subseteq Z \cup Z'$ where $Z' \subseteq Y$. Thus $|D| \leq |Z| + |Z'|$ which implies $\gamma(U_2(G)) \leq \binom{n}{2} - \left(1 + \binom{\Delta}{2}\right)$.
Equality is attained for $G \cong K_3, K_4$, triangle with a tail attached to one of its vertex. $\qquad\square$

Corollary 2.11. *If a graph G has atleast one full degree vertex then $\gamma(U_2(G)) \leq n - 2$. Equality is attained for $G \cong K_{1,n}$, K_4, $K_4 - e$, triangle with a tail attached to one of its vertex.*

Theorem 2.12. *For a graph G, $\gamma(\overline{U_2(G)}) \leq 3$.*

Proof. Consider the vertex $\{v_1, v_2\}$ of $\overline{U_2(G)}$ which dominates $\binom{n-2}{2}$ vertices of the type $\{\{v_h, v_k\} : 3 \leq h \leq n - 1; h + 1 \leq k \leq n\}$ in $\overline{U_2(G)}$. Then of the remaining $2n - 4$ vertices, the vertex $\{v_1, v_3\}$ dominates vertices of the type $\{\{v_2, v_k\} : 4 \leq k \leq n\}$ and the vertex $\{v_2, v_3\}$ dominates $\{\{v_1, v_k\} : 4 \leq k \leq n\}$ covering $\overline{U_2(G)}$ which implies $\gamma(\overline{U_2(G)}) \leq 3$.
Equality is attained for $G \cong K_n$. $\qquad\square$

Theorem 2.13. *$\gamma(\overline{U_2(G)}) = 1$ if and only if atleast one component of G is isomorphic to K_2 or G has atleast two isolates.*

Theorem 2.14. *If any graph G has atleast one pendant vertex then $\gamma(\overline{U_2(G)}) = 2$.*

3. Conclusion

Studying graph products help us to analyze networks which are constructed by imposing some constraints on the original networks. Double vertex graph is one such graph product. We have obtained the bounds for the domination number of double vertex graph and its complement in terms of parameters of G. For future work, based on practical requirements different types of domination can be studied.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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