Journal of Informatics and Mathematical Sciences
Volume 1 (2009), Numbers 2 & 3, pp. 91–97

(*Invited paper*)

Inverse Eigenvalue Problem with Non-simple Eigenvalues for Damped Vibration Systems*

M. Mohseni Moghadam and A. Tajaddini

Abstract. In this paper, we will present a general form of real and symmetric $n \times n$ matrices M, C and K for a quadratic inverse eigenvalue problem QIEP: $Q(\lambda) \equiv (\lambda^2 M + \lambda C + K)x = 0$, so that $Q(\lambda)$ has a prescribed set of k eigenvalues with algebraic multiplicity n_i, $i = 1, \cdots, k$ (which $2n_1 + 2n_2 + \cdots + 2n_l + n_{l+1} + \cdots + n_k = 2n$). This paper generalizes the method of inverse problem for self-adjoint linear pencils, to self-adjoint quadratic pencils $Q(\lambda)$. It is shown that this inverse problem involves certain free parameters. Via appropriate choice of free variables in the general form of QIEP, we solve a QIEP.

1. Introduction

The problem of finding scalars $\lambda \in \mathscr{C}$ and nontrivial vector $x \in \mathscr{C}^n$ such that

$$Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0, \tag{1.1}$$

where M, C and K are given $n \times n$ real matrices, is known as the quadratic eigenvalue problem QEP. The nonzero vectors x and the corresponding scalars λ are called eigenvectors and eigenvalues of the QEP, respectively. It is known that if the leading coefficient matrix M is nonsingular, then the quadratic pencil will have $2n$ eigenvalues over $\mathscr{C}$.

A theoretical analysis of QEPs can be found in the book by written Gohberg, Lancaster and Rodman [5]. Many applications, mathematical properties and a variety of numerical techniques for the QEP are surveyed in the treatise by Tisseur and Meerbergen [4].

Generally, in many mathematical modeling, there is a correspondence between the internal parameters and the external behavior of a system from a priori known physical parameters such as mass, length, elasticity, inductance, capacitance, and

2000 *Mathematics Subject Classification.* 15-XX; 15-02; 15A29.
Key words and phrases. Inverse problem; Quadratic form; Non-simple eigenvalues.
**This work has been partially supported by the Center of Excellence of Linear Algebra and Optimization, Shahid Bahonar University of Kerman.*

so on, is referred to a direct problem. In contrast, the inverse problem is to determine or estimate the parameters of the system according to its measured or expected behavior. The concern in the direct problem is to express the behavior in term of parameters whereas in the inverse problem concern is to express the parameters in terms of behavior. The inverse problem is as important as the direct problem in applications.

The purpose of this paper is to make use of a parametrization method of inverse eigenvalue problems for self-adjoint linear pencils, that Chu, Datta, Lin and Xu developed earlier [1], to self-adjoint quadratic pencils $Q(\lambda)$. Recently, Kuo, Lin and Xu [2], Chu and Xu [3] proposed to parametrize a self-adjoint inverse eigenvalue problem.

We shall limit a quadratic inverse eigenvalue problem QIEP to the following realm.

Determine a general solution of real symmetric coefficients matrices M, C and K so that the resulting QEP has a prescribed set of an eigenpair.More precisely, by a QIEP we refer to the following inverse problem:

QIEP: Let $(\Lambda, X) \in R^{2n\times 2n} \times R^{n\times 2n}$ be a given pair of matrices, where

$$\Lambda = \mathrm{diag}\{\lambda_1^{[2]}, \ldots, \lambda_l^{[2]}, \lambda_{l+1} I_{nl+1}, \ldots, \lambda_k I_{nk}\}, \tag{1.2}$$

$$\lambda_i^{[2]} = \begin{pmatrix} \alpha_i I_{n_i} & \beta_i I_{n_i} \\ -\beta_i I_{n_i} & \alpha_i I_{n_i} \end{pmatrix}, \quad i = 1, \ldots, l, \tag{1.3}$$

and

$$X = [X_{1R}, X_{1I}, \ldots, X_{lR}, X_{lI}, X_{l+1}, \ldots, X_k], \tag{1.4}$$

where n_i, $i = 1, \cdots, k$ are algebraic multiplicity corresponding to λ_i and $X_{iR}, X_{iI} \in R^{n\times n_i}$, $i = 1, \ldots, l$, and $X_i \in R^{n\times n_i}$, $i = l+1, \ldots, k$, and so $\begin{pmatrix} X \\ X\Lambda \end{pmatrix}$ is nonsingular. The true eigenvalues and eigenvectors are readily identifiable by the transformation

$$R := \mathrm{diag}\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} I_{n1} & I_{n1} \\ iI_{n1} & -iI_{n1} \end{pmatrix}, \cdots, \frac{1}{\sqrt{2}} \begin{pmatrix} I_{nl} & I_{nl} \\ iI_{nl} & -iI_{nl} \end{pmatrix}, I_{nl+1}, \cdots, I_{nk} \right\}, \tag{1.5}$$

with $i = \sqrt{-1}$. That is, by defining

$$\begin{aligned} \tilde{\Lambda} &= R^H \Lambda R \\ &= \mathrm{diag}\{\lambda_1 I_{n1}, \lambda_2 I_{n1}, \cdots, \lambda_{2l-1} I_{nl}, \lambda_{2l} I_{nl}, \lambda_{2l+1} I_{nl+1}, \cdots, \lambda_k I_{nk}\} \in \mathscr{C}^{2n\times 2n}, \end{aligned} \tag{1.6}$$

$$\tilde{X} = XR = [X_1, X_2, \cdots, X_{2l-1}, X_{2l}, X_{2l+1}, \cdots, X_k] \in \mathscr{C}^{n\times 2n}. \tag{1.7}$$

The true (complex-valued) eigenvalues and eigenvectors of the desired quadratic pencil $Q(\lambda)$ can be induced from the pair (Λ, X) of real matrices. In this case, note that $X_{2j-1} = X_{jR} + iX_{jI}$, $X_{2j} = X_{jR} - iX_{jI}$, $\lambda_{2j-1} = \alpha_j + i\beta_j$, and $\lambda_{2j} = \alpha_j - i\beta_j$, for $j = 1, \cdots, l$, where as X_j and λ_j are all real-valued for $j = 2l+1, \cdots, k$. Find a general form for symmetric matrices M, C and K which satisfy in the following eqnarray

$$MX\Lambda^2 + CX\Lambda + KX = 0. \tag{1.8}$$

Now, we define a standard eigenpair for the quadratic pencil that introduced in [5].

Definition 1.1. A pair of matrices $(\Lambda, X) \in R^{2n\times 2n} \times R^{n\times 2n}$ is called a standard eigenpair for the quadratic pencil $Q(\lambda)$ if and only if the matrix

$$Y := \begin{pmatrix} X \\ X\Lambda \end{pmatrix} \tag{1.9}$$

is nonsingular and the eqnarray

$$MX\Lambda^2 + CX\Lambda + KX = 0$$

holds.

2. **Solving IQEP**

In this section, we shall solve the IQEP for a given matrix pair $(\Lambda, X) \in R^{2n\times 2n} \times R^{n\times 2n}$ as in (1.2) and (1.4) under assumption $\begin{pmatrix} X \\ X\Lambda \end{pmatrix}$ is nonsingular. It is easy to see that $Q(\lambda)x = 0$ if and only if

$$L(\lambda)\begin{pmatrix} x \\ \lambda x \end{pmatrix} = 0, \tag{2.1}$$

where

$$L(\lambda) := \lambda \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} - \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix}. \tag{2.2}$$

We want to construct a general form for M, C and K. We first tell the theorem that Chu, Datta, Lin and Xu [1] prove for self-adjoint linear pencils.

Theorem 2.1. *A self-adjoint linear pencil can have arbitrary eigenstructure with distinct eigenvalues and linearly independent eigenvectors. Indeed, given an eigenstructure (X, Λ), the solutions (A, B) form a subspace of dimensionality n in the product space $R^{n\times n} \times R^{n\times n}$ and can be parametrized by the diagonal matrix Γ via the following relationships*

$$A = X^{-T}\Gamma X^{-1},$$
$$B = X^{-T}\Gamma\Lambda X^{-1}.$$

We consider the problem quadratic that its eigenvalues are non-simple and so $Y = \begin{pmatrix} X \\ X\Lambda \end{pmatrix}$ is nonsingular. In order to, we construct a general form for matrices M, C and K, we make use of Lancaster linearization and Chu, Datta, Lin and Xu's proof idea.

We know that $\left(\Lambda, Y = \begin{pmatrix} X \\ X\Lambda \end{pmatrix}\right)$ is an eigenpair for linear pencil $L(\lambda)$. Let us consider the linear pencil $\lambda A - B$ where

$$A = \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \tag{2.3}$$

and

$$B = \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix}. \tag{2.4}$$

For the linear pencil $\lambda A - B$ to have eigenstructure (Λ, Y), it is necessary that

$$\begin{pmatrix} -I_{2n} & \Lambda_{2n}^T \end{pmatrix} \begin{pmatrix} Y^T B^T \\ Y^T A^T \end{pmatrix} = 0. \tag{2.5}$$

On the other hand, it is trivial that

$$\begin{pmatrix} -I_{2n} & \Lambda_{2n}^T \end{pmatrix} \begin{pmatrix} \Lambda^T S \\ S \end{pmatrix} = 0, \tag{2.6}$$

for any $S \in R^{2n \times 2n}$. Thus we obtain a parametric representation

$$A = S^T Y^{-1}, \tag{2.7}$$

$$B = S^T \Lambda Y^{-1}. \tag{2.8}$$

We are interested in selecting S so as to construct self-adjoint pencils. For matrix A to be symmetric, the matrix S must be such that

$$S^T Y^{-1} = Y^{-T} S$$

implying that the matrix Γ defined by

$$\Gamma := Y^T S^T = SY, \tag{2.9}$$

and its inverse are symmetric. For matrix B to be symmetric, the matrix S must also be such that

$$S^T \Lambda Y^{-1} = Y^{-T} \Lambda^T S,$$

implying that

$$\Gamma^{-1} \Lambda^T = \Lambda \Gamma^{-1}. \tag{2.10}$$

Partition Γ^{-1} according to the sizes of sub-matrices in Λ, we will have:

$$\Gamma^{-1} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1l} & \cdots & \Gamma_{1k} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2l} & \cdots & \Gamma_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Gamma_{k-11} & \Gamma_{k-12} & \cdots & \Gamma_{k-1l} & \cdots & \Gamma_{k-1k} \\ \Gamma_{k1} & \Gamma_{k2} & \cdots & \Gamma_{kl} & \cdots & \Gamma_{kk} \end{pmatrix}, \tag{2.11}$$

where Γ_{ii} is $2n_i \times 2n_i$ matrix, $i = 1, 2, \ldots, l$, and Γ_{ii} is $n_i \times n_i$ matrix , $i = l+1, \ldots, k$. Substituting (2.11), (1.2) into (2.10), we obtain

$$\begin{aligned} \Gamma_{ij}(\lambda_j^{[2]})^T &= \lambda_i^{[2]} \Gamma_{ij} \quad \text{if } 1 \le i, j \le l, \\ \Gamma_{ij}\lambda_j &= \lambda_i^{[2]} \Gamma_{ij} \quad \text{if } 1 \le i \le l, l+1 \le j \le k, \\ \Gamma_{ij}\lambda_j &= \lambda_i \Gamma_{ij} \quad \text{if } l+1 \le i, j \le k. \end{aligned} \tag{2.12}$$

First let us consider the case $1 \le i, j \le l$. Without loss of generality, if we write

$$\Gamma_{ij} = \begin{pmatrix} U & W \\ W & V \end{pmatrix}$$

then

$$\begin{pmatrix} (\alpha_i - \alpha_j)U + (\beta_i - \beta_j)W & -\beta_i U + (\alpha_i - \alpha_j)W - \beta_j V \\ \beta_j U + (\alpha_i - \alpha_j)W + \beta_i V & (\beta_i - \beta_j)W - (\alpha_i - \alpha_j)V \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that if $i=j$ then

$$\Gamma_{ii}=\begin{pmatrix} U_i & W_i \\ W_i & -U_i \end{pmatrix},$$

where U_i and W_i are symmetric, give rise $n_i(n_i+1)$ degree of freedom.If $i\neq j$, $\Gamma_{ij}=0$. If $1\leq i\leq l$, $l+1\leq j\leq k$, then Γ_{ij} is a 2×1 block matrix and is equal to zero. Finally for $l+1\leq i,j\leq k$, if $i\neq j$, then $\Gamma_{ij}=0$, otherwise, $\Gamma_{ii}=U_i$ is arbitrary. Thus, we conclude that the symmetric matrix Γ^{-1} is equal to the following block-diagonal

$$\Gamma^{-1}=\mathrm{diag}\left\{\begin{pmatrix} U_1 & W_1 \\ W_1 & -U_1 \end{pmatrix},\cdots,\begin{pmatrix} U_l & W_l \\ W_l & -U_l \end{pmatrix},U_{l+1},\cdots,U_k\right\}, \tag{2.13}$$

where U_i, $i=1,\cdots,k$, and W_i, $i=1,\cdots,l$ are symmetric. Upon choosing an arbitrary Γ, a substitution by $S^T=Y^{-T}\Gamma$ implies that the pencil $\lambda A-B$ with

$$A=Y^{-T}\Gamma Y^{-1}, \tag{2.14}$$

and

$$B=Y^{-T}\Gamma\Lambda Y^{-1} \tag{2.15}$$

is self-adjoint and has eigenstructure (Λ,Y).

Moreover matrix A is invertible, and its inverse matrix is

$$A^{-1}=\begin{pmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}CM^{-1} \end{pmatrix}. \tag{2.16}$$

We set

$$A^{-1}=Y\Gamma^{-1}Y^T. \tag{2.17}$$

Substituting (2.16), (1.9) into (2.17), we have

$$M=(X\Gamma^{-1}\Lambda^TX^T)^{-1}, \tag{2.18}$$

$$C=-MX\Lambda\Gamma^{-1}\Lambda^TX^TM, \tag{2.19}$$

and

$$X\Gamma^{-1}X^T=0. \tag{2.20}$$

Post-multiplying (1.8) by $\Gamma^{-1}\Lambda^TX^T$ and applying (2.18) and (2.19), we obtain

$$K=-MX\Lambda^3\Gamma^{-1}X^TM+CM^{-1}C. \tag{2.21}$$

We thus declare the following theorem.

Theorem 2.2. *Let a standard eigenpair* $(\Lambda,X)\in R^{2n\times 2n}\times R^{n\times 2n}$ *as* (1.2), (1.4)*, be given then the general solution of QIEP forms are as:*

$$M=(X\Gamma^{-1}\Lambda^TX^T)^{-1},$$

$$C=-MX\Lambda\Gamma^{-1}\Lambda^TX^TM,$$

and

$$K=-MX\Lambda^3\Gamma^{-1}X^TM+CM^{-1}C,$$

where Γ^{-1} *is given by* (2.13)*, and* $X\Gamma^{-1}X^T=0$.

The following result can be regarded as the converse of Theorem 2.2.

Theorem 2.3. *Let* $(\Lambda, X) \in R^{2n\times 2n} \times R^{n\times 2n}$ *be given matrices as in* (1.2), (1.4). *suppose there exists a symmetric and nonsingular matrix* $\Gamma^{-1} \in R^{2n\times 2n}$ *such that* $X\Gamma^{-1}\Lambda^T X^T$ *is nonsingular,* (2.20) *and* (2.10) *hold, then* $\begin{pmatrix} X \\ X\Lambda \end{pmatrix}$ *is nonsingular and the eqnarray* (1.8) *holds for the self-adjoint quadratic pencil* $Q(\lambda)$ *whose matrix coefficients* M, C, *and* K *are defined according to* (2.18), (2.19), (2.21), *respectively. That is,* (Λ, X) *is a standard eigenpair for* $Q(\lambda)$.

Proof. Since $X\Gamma^{-1}\Lambda^T X^T$ is nonsingular, M can be defined. We want to show that $\begin{pmatrix} X \\ X\Lambda \end{pmatrix}$ in invertible. By substitution (2.16), (2.2) into (2.17), we have

$$\begin{pmatrix} X\Gamma^{-1}X^T \\ X\Lambda\Gamma^{-1}X^T \end{pmatrix} = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}, \tag{2.22}$$

and

$$\begin{pmatrix} X\Gamma^{-1}\Lambda^T X^T \\ X\Lambda\Gamma^{-1}\Lambda^T X^T \end{pmatrix} = \begin{pmatrix} M^{-1} \\ -M^{-1}CM^{-1} \end{pmatrix}. \tag{2.23}$$

Hence (2.22) implies

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix} \Gamma^{-1}X^T M = \begin{pmatrix} 0 \\ I_n \end{pmatrix}. \tag{2.24}$$

Also (2.23) implies

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix} \Gamma^{-1}\Lambda^T X^T = \begin{pmatrix} I_n \\ -M^{-1}C \end{pmatrix} M^{-1}. \tag{2.25}$$

post-multiplying (2.25) by M and applying (2.24), we obtain

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix} (\Gamma^{-1}\Lambda^T X^T M + \Gamma^{-1}X^T C) = \begin{pmatrix} I_n \\ 0 \end{pmatrix}. \tag{2.26}$$

By (2.24), (2.26), we observe that

$$\begin{pmatrix} X \\ X\Lambda \end{pmatrix} \begin{pmatrix} \Gamma^{-1}\Lambda^T X^T M + \Gamma^{-1}X^T C & \Gamma^{-1}X^T M \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix},$$

implies that the matrix $\begin{pmatrix} X \\ X\Lambda \end{pmatrix}$ is nonsingular. It follows that

$$\begin{aligned} MX\Lambda^2 \begin{pmatrix} X \\ X\Lambda \end{pmatrix}^{-1} &= MX\Lambda^2 \begin{pmatrix} \Gamma^{-1}\Lambda^T X^T M + \Gamma^{-1}X^T M & \Gamma^{-1}X^T M \end{pmatrix} \\ &= \begin{pmatrix} -K & -C \end{pmatrix}, \end{aligned}$$

which is equivalent to $MX\Lambda^2 + CX\Lambda + KX = 0$.

3. **Conclusion**

In this paper, we mainly present a general solution of an QIEP with a prescribed standard eigenpair $(\Lambda, X) \in R^{2n\times 2n} \times R^{n\times 2n}$. Via appropriate choice of free variables in the general solution we can solve a quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ with M, C and K being symmetric.How to choice the total degree of freedoms in the general form of QIEP to obtain more exact quadratic pencil $Q(\lambda)$.

References

[1] M.T. Chu, B.N. Datta, W.-W. Lin and S.-F. Xu, The spill-over phenomenon in quadratic model updating, *AIAA Journal* **46** (2008), 420–428.

[2] Y.-C. Kuo, W.-W. Lin and S.-F. Xu, On a general solution of partially described inverse quadratic eigenvalue problems and its applications, NCTS Technical Report 2005-1-2, National Tsing-Hua Univ., Taiwan. `http://math.cts.nthu.edu.tw/Mathematics/preprints/preprints/prep2005-1-001`.

[3] M.T. Chu and S-F. Xu, Spectral decomposition of self-adjoint quadratic pencils and its applications, *Math. Comput.* **78** (2009), 293–313.

[4] T. Francoise and M. Karl, The quadratic eigenvalue problem, *SIAM Review* **43** (2) (2001), 235–286.

[5] I. Gohberge, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, Inc., New York – London, 1982.

[6] D.J. Inman and A.N. Andry, Jr., Some results on the nature of eigenvalues of discrete damped linear systems, *Journal of Applied Mechanics* **47**(1980), 927–930.

[7] P. Lancaster, *Inverse spectral problems for semi-simple problems damped vibrating systems*, *SIAM J. Matrix Anal. Appl.* **29**(2007), 279–301.

M. Mohseni Moghadam, *Mahani Mathematical Research Center, Kerman 76169-14111, Iran.*
E-mail: `Mohseni@mail.uk.ac.ir`

A. Tajaddini, *Department of Mathematics, Shahid Bahonar university of Kerman, Kerman 76169-14111, Iran.*
E-mail: `Azita_Tajaddinii@yahoo.com`

Received May 18, 2009
Accepted July 1, 2009