Journal of Informatics and Mathematical Sciences
Vol. 12, No. 1, pp. 1–14, 2020
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v12i1.1088

http://www.rgnpublications.com

Research Article

Right Semi-Tensor Product for Matrices Over a Commutative Semiring

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Abstract. This paper generalizes the right semi-tensor product for real matrices to that for matrices over an arbitrary commutative semiring, and investigates its properties. This product is defined for any pair of matrices satisfying the matching-dimension condition. In particular, the usual matrix product and the scalar multiplication are its special cases. The right semi-tensor product turns out to be an associative bilinear map that is compatible with the transposition and the inversion. The product also satisfies certain identity-like properties and preserves some structural properties of matrices. We can convert between the right semi-tensor product of two matrices and the left semi-tensor product using commutation matrices. Moreover, certain vectorizations of the usual product of matrices can be written in terms of the right semi-tensor product.

Keywords. Right semi-tensor product; Kronecker product; Commutative semiring; Vector operator; Commutation matrix

MSC. 15A69; 15B33; 16Y60

Received: March 17, 2019 **Accepted:** August 16, 2019

1. Introduction

The theory of matrices whose entries come from a ring-like structure such as a (commutative) semiring is of interest (see e.g. [16, 21, 22, 25]). Applications of this theory arise in many areas, including combinatorics, optimization, operation research, information sciences, and control engineering (see e.g. [14, 17, 18, 24]). From the prospective of scientific computing, a rectangular

matrix is a two-dimensional array for stacking data, and a matrix product is a way to produce a new data. Certain matrix products have been investigated for matrices over a commutative semiring, e.g. tensor product [23], box product [3], Hadamard product [5], and block Hadamard product [5].

The left semi-tensor product $\ltimes$, introduced by D. Cheng [6], is a generalization of the usual matrix product; e.g. [7, 8, 10]. The left semi-tensor product of two real matrices A and B of dimensions $m \times n$ and $p \times q$ is defined when the pair (A,B) satisfies the matching-dimension condition, i.e. n divides p or p divides n. In particular, we have $A \ltimes B = AB$ when $n = p$. This product has a strong relationship to the tensor product, and thus its name. We can apply the left semi-tensor product to treat multi-dimensional data and to transform certain nonlinear problems into linear ones. This matrix product has several applications in mathematical logic, abstract algebra, Boolean networks and so on (see e.g. [8, 9, 11, 12, 20, 24]). Recently, the left semi-tensor product of matrices over an arbitrary commutative semiring was invesigated in [4]. In this case, the product is associative and satisfies certain identity-like properties. Moreover, it is compatible with certain matrix operations, namely, the addition, the scalar multiplication, the transposition, and the inversion.

The right semi-tensor product, introduced by D. Cheng [8], is another generalization of the usual matrix product. The right semi-tensor product is naturally defined through certain formulas related to tensor products.

In the present paper, we provide a natural definition of the right semi-tensor product for matrices over a commutative semiring satisfying the matching-dimension condition. This definition is given in terms of a certain formula involving the tensor product (see Section 3). Then, we show that this matrix product is associative, bilinear and satisfies identity-like properties. Similar to the usual product, this product is also compatible with the transposition, the inversion, the tensor product, and traces. After that, we discuss relations of this product to certain operations for data rearrangement, namely, row/column vector operators and swapping operators (commutation matrices) (see Section 4).

2. Preliminaries

In this section, we supply preliminaries on matrices over a commutative semiring, the tensor product, and the left semi-tensor product.

2.1 Matrices Over a Commutative Semiring

Zimmerman [26], and Golan [17] introduced the following definition.

Definition 1. A commutative semiring is a 5-tuple $(L,+,\cdot,0,1)$ consisting of a set L with two distinguish elements 0 and 1 together with two binary operations, called the addition $(+)$ and the multiplication $(\cdot)$, such that

(i) $(L,+,0)$ is a commutative monoid with $0\cdot a = 0$ for all $a \in L$;

(ii) $(L,\cdot,1)$ is a commutative monoid;

(iii) the multiplication is distributive over the addition.

The sets $[0,\infty)$, $\mathbb{Z}_n$, and every field are commutative semirings with respect to usual operations. The Cartesian product of two commutative semirings is again a commutative semiring with respect to pointwise operations.

Example 1. More interesting examples of commutative semirings are as follows:

(i) the fuzzy algebra $[0,1]$ under the max/min operations (see e.g. [19]).

(ii) the nonnegative integers under the operations of gcd (greatest common divisor) and lcm (least common multiple).

(iii) the schedule algebra (or max-plus algebra) $\mathbb{R}\cup\{-\infty\}$ under the max operation and the addition (see e.g. [1, 15]).

(iv) the extended positive half-line $[0,\infty]$ under the max/min operations.

Example 2. Recall that an MV-algebra (see e.g. [2, 13]) is a 6-tuple $(L,\oplus,\odot,\neg,0,1)$ consisting of a set L with two distinct elements $0,1\in L$ together with two binary operations $\oplus$ and $\odot$, and a unary operation $\neg$ on L such that the following properties hold for all $x,y,z\in L$:

$$(x\oplus y)\oplus z = x\oplus(y\oplus z),\quad x\oplus y = y\oplus x,\quad x\oplus 0 = x,\quad x\oplus 1 = 1,$$

$$\neg\neg x = x,\quad \neg 0 = 1,\quad x\odot y = \neg(\neg x\oplus\neg y),\quad \neg(\neg x\oplus y)\oplus y = \neg(\neg y\oplus x)\oplus x.$$

We put $x\vee y = (x\odot\neg y)\oplus y$ and $x\wedge y = (x\oplus\neg y)\odot y$ for each $x,y\in L$. Then $(L,\vee,\odot,0,1)$ and $(L,\wedge,\oplus,1,0)$ are commutative semirings.

Throughout this paper, let $\mathfrak{S}$ be a commutative semiring. For each $m,n\in\mathbb{N}$, denote by $\mathscr{M}_{m,n}(\mathfrak{S})$ the set of m-by-n matrices over $\mathfrak{S}$. We use $\mathrm{Row}_i(A)$ for the ith row of a matrix A and $\mathrm{Col}_i(A)$ for the ith column of A. The existence of 0 and 1 allows us to define the identity matrix, denoted by I or I_n for the identity matrix of size $n\times n$. We define the addition, the scalar multiplication, the usual multiplication, the transposition, and the trace for matrices as in the usual ways for real matrices. It turns out such matrix operations satisfy usual properties for those of real matrices, except for properties involving additive/multiplicative inverses. In particular, we have:

Proposition 1 ([22]). *If $A,B\in\mathscr{M}_{n,n}(\mathfrak{S})$ are such that $AB = I_n$, then $BA = I_n$.*

A matrix $A\in\mathscr{M}_{n,n}(\mathfrak{S})$ is said to be *invertible* if there is a matrix $B\in\mathscr{M}_{n,n}(\mathfrak{S})$ such that $AB = I_n = BA$, or equivalently, $AB = I_n$ (by Proposition 1). If such B exists, it is uniquely determined, and we write $B = A^{-1}$. A matrix $A\in\mathscr{M}_{n,n}(\mathfrak{S})$ is said to be *similar* to $B\in\mathscr{M}_{n,n}(\mathfrak{S})$ if there is an invertible matrix S such that $S^{-1}AS = B$, written $A\sim B$. A matrix $A\in\mathscr{M}_{n,n}(\mathfrak{S})$ is said to be *orthogonal* if $A^TA = I_n = AA^T$, or equivalently, $A^TA = I_n$.

The following convention will be often used in the paper.

Definition 2. Let $A\in\mathscr{M}_{m,n}(\mathfrak{S})$ and $B\in\mathscr{M}_{p,q}(\mathfrak{S})$. If $n = pt$ for some $t\in\mathbb{N}$, then we write $A\succ_t B$ or $A\succ B$. If $nt = p$ for some $t\in\mathbb{N}$, we write $A\prec_t B$ or $A\prec B$. In both cases, we say that the ordered pair (A,B) satisfies the *matching-dimension condition*.

2.2 The Tensor Product of Matrices Over a Commutative Semiring

Definition 3. The *tensor product* or the *Kronecker product* of $A = [a_{ij}] \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{p,q}(\mathbb{S})$ is defined by $A \otimes B = \left[a_{ij}B\right]_{ij} \in \mathscr{M}_{mp,nq}(\mathbb{S})$. That is, each (i,j)th block of $A \otimes B$ is given by $a_{ij}B$ for each i,j.

Fundamental properties of the tensor product are listed below.

Lemma 1 ([23]). *The following properties hold for all matrices over $\mathbb{S}$ with appropriate sizes:*

1. $A \otimes (B+C) = A \otimes B + A \otimes C$ *and* $(B+C) \otimes A = B \otimes A + C \otimes A$,
2. $(kA) \otimes B = k(A \otimes B) = A \otimes (kB)$ *for any* $k \in \mathbb{S}$,
3. $(A \otimes B)^T = A^T \otimes B^T$,
4. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,
5. *if A and B are invertible, then $A \otimes B$ is invertible with* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

2.3 The Left Semi-Tensor Product of Matrices Over a Commutative Semiring

First, we consider the left semi-tensor product of a row vector $X \in \mathscr{M}_{1,m}(\mathbb{S})$ and a column vector $Y \in \mathscr{M}_{p,1}(\mathbb{S})$.

(i) For the case $X \succ_t Y$, we split X into p equal-size blocks as $\left(X^1, X^2, \ldots, X^p\right)$, such that $X^i \in \mathscr{M}_{1,t}(\mathbb{S})$ for all $i = 1, \ldots, p$. We define

$$X \ltimes Y = \sum_{i=1}^{p} y_i X^i \in M_{1,t}(\mathbb{S}). \tag{1}$$

(ii) For the case $X \prec_t Y$, we split Y into m equal-size blocks as $\left(Y^1, Y^2, \ldots, Y^m\right)$, such that $Y^i \in \mathscr{M}_{t,1}(\mathbb{S})$ for all $i = 1, \ldots, m$. We define

$$X \ltimes Y = \sum_{i=1}^{m} x_i Y^i \in M_{t,1}(\mathbb{S}). \tag{2}$$

Definition 4. Let $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in M_{p,q}(\mathbb{S})$ be such that the pair (A,B) satisfies the matching-dimension condition. The *left semi-tensor product* of A and B is defined as

$$A \ltimes B = \begin{bmatrix} \mathrm{Row}_1(A) \ltimes \mathrm{Col}_1(B) & \mathrm{Row}_1(A) \ltimes \mathrm{Col}_2(B) & \cdots & \mathrm{Row}_1(A) \ltimes \mathrm{Col}_q(B) \\ \mathrm{Row}_2(A) \ltimes \mathrm{Col}_1(B) & \mathrm{Row}_2(A) \ltimes \mathrm{Col}_2(B) & \cdots & \mathrm{Row}_2(A) \ltimes \mathrm{Col}_q(B) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A) \ltimes \mathrm{Col}_1(B) & \mathrm{Row}_m(A) \ltimes \mathrm{Col}_2(B) & \cdots & \mathrm{Row}_m(A) \ltimes \mathrm{Col}_q(B) \end{bmatrix}.$$

One of the most remarkable properties of the left semi-tensor product is the following:

Proposition 2 ([4]). *If $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{nt,q}(\mathbb{S})$, then $A \ltimes B = (A \otimes I_t)B$.*
If $A \in \mathscr{M}_{m,nt}(\mathbb{S})$ and $B \in \mathscr{M}_{n,q}(\mathbb{S})$, then $A \ltimes B = A(B \otimes I_t)$.

This proposition establishes a strong relation between the tensor product and the left semi-tensor product. Thus it justifies the name “semi-tensor product".

Proposition 3 ([4]). *Let $A \in \mathscr{M}_{m,n}(\mathbb{S})$, $B \in \mathscr{M}_{p,q}(\mathbb{S})$, $X \in \mathscr{M}_{n,1}(\mathbb{S})$, $Y \in \mathscr{M}_{q,1}(\mathbb{S})$, $R \in \mathscr{M}_{1,m}(\mathbb{S})$ and $S \in \mathscr{M}_{1,p}(\mathbb{S})$. Then*

$$AX \ltimes BY = (A \otimes B)(X \ltimes Y), \qquad RA \ltimes SB = (R \ltimes S)(B \otimes A).$$

3. Fundamentals Properties of the Right Semi-Tensor Product of Matrices Over a Commutative Semiring

In this section, we define the right semi-tensor product of matrices over a commutative semiring for any pair of matrices that satisfied the matching dimension condition. Then, we investigate its properties related to matrix operations.

From Proposition 2, it is natural to define the right semi-tensor product as follows.

Definition 5. We define the *right semi-tensor product* of $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{nt,q}(\mathbb{S})$ by

$$A \rtimes B = (I_t \otimes A)B \in \mathscr{M}_{mt,q}(\mathbb{S}).$$

When $A \in \mathscr{M}_{m,nt}(\mathbb{S})$ and $B \in \mathscr{M}_{n,q}(\mathbb{S})$, its right semi-tensor product is defined by

$$A \rtimes B = A(I_t \otimes B) \in \mathscr{M}_{m,qt}(\mathbb{S}).$$

Remark 1. If $X \in \mathscr{M}_{n,1}(\mathbb{S})$ and $Y \in \mathscr{M}_{q,1}(\mathbb{S})$ are column vectors, then $X \rtimes Y = Y \otimes X$. If $R \in \mathscr{M}_{1,m}(\mathbb{S})$ and $S \in \mathscr{M}_{1,p}(\mathbb{S})$ are row vectors, then $X \rtimes Y = X \otimes Y$.

Remark 2. Explicit formulas of the right semi-tensor product between a row vector and a column vector are as follows:
For the case that $A \in \mathscr{M}_{1,n}(\mathbb{S})$ and $B \in \mathscr{M}_{nt,1}(\mathbb{S})$, using block-matrix multiplication we have

$$A \rtimes B = (I_t \otimes A)B = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \begin{bmatrix} B^1 \\ \vdots \\ B^t \end{bmatrix} = \begin{bmatrix} AB^1 \\ \vdots \\ AB^t \end{bmatrix},$$

here each block B^i consists of n rows of B.
For the case that $A \in \mathscr{M}_{1,nt}(\mathbb{S})$ and $B \in \mathscr{M}_{n,1}(\mathbb{S})$, we have

$$A \rtimes B = A(I_t \otimes B) = \begin{bmatrix} A^1 & \cdots & A^t \end{bmatrix} \begin{bmatrix} B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B \end{bmatrix} = \begin{bmatrix} A^1B & \cdots & A^tB \end{bmatrix}.$$

Remark 3. We shall compare Definition 4 to Definition 5. First, consider $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{nt,q}(\mathbb{S})$ in which

$$B = \begin{bmatrix} B_1^1 & \cdots & B_q^1 \\ \vdots & \ddots & \vdots \\ B_1^t & \cdots & B_q^t \end{bmatrix},$$

where $B_j^i \in \mathscr{M}_{n,1}(\mathbb{S})$ for each i, j. Then

$$A \rtimes B = (I_t \otimes A)B = \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix} \begin{bmatrix} B_1^1 & \cdots & B_q^1 \\ \vdots & \ddots & \vdots \\ B_1^t & \cdots & B_q^t \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Row}_1(A)B_1^1 & \cdots & \mathrm{Row}_1(A)B_q^1 \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A)B_1^1 & \cdots & \mathrm{Row}_m(A)B_q^1 \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_1(A)B_1^t & \cdots & \mathrm{Row}_1(A)B_q^t \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A)B_1^t & \cdots & \mathrm{Row}_m(A)B_q^t \end{bmatrix}.$$

Now, consider the case that $A \in \mathscr{M}_{m,nt}(\mathfrak{S})$ and $B \in \mathscr{M}_{n,q}(\mathfrak{S})$. We have

$$\begin{aligned} A \rtimes B &= A(I_t \otimes B) \\ &= \begin{bmatrix} A_1^1 & \cdots & A_q^1 \\ \vdots & \ddots & \vdots \\ A_1^t & \cdots & A_q^t \end{bmatrix} \begin{bmatrix} B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Row}_1(A) \rtimes B \\ \vdots \\ \mathrm{Row}_m(A) \rtimes B \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Row}_1(A) \rtimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_1(A) \rtimes \mathrm{Col}_q(B) \\ \vdots & \ddots & \vdots \\ \mathrm{Row}_m(A) \rtimes \mathrm{Col}_1(B) & \cdots & \mathrm{Row}_m(A) \rtimes \mathrm{Col}_q(B) \end{bmatrix}. \end{aligned}$$

Remark 4. If $A \in \mathscr{M}_{m,n}(\mathfrak{S})$ and $B \in \mathscr{M}_{n,q}(\mathfrak{S})$, then $A \rtimes B = AB$. If $k \in \mathfrak{S}$ and $A \in \mathscr{M}_{m,n}(\mathfrak{S})$, then

$$[k] \rtimes A = (I_m \otimes [k])A = kI_m A = kA.$$

Thus, the right semi-tensor product includes the conventional product and the scalar multiplication as special cases.

The next result is a parallel result of Proposition 3. Its shows relations between the left/right semi-tensor products, the tensor product, and the usual product.

Proposition 4. *Let $A \in \mathscr{M}_{m,n}(\mathfrak{S})$, $B \in \mathscr{M}_{p,q}(\mathfrak{S})$, $X \in \mathscr{M}_{n,1}(\mathfrak{S})$, $Y \in \mathscr{M}_{q,1}(\mathfrak{S})$, $R \in \mathscr{M}_{1,m}(\mathfrak{S})$ and $S \in \mathscr{M}_{1,p}(\mathfrak{S})$. Then*

$$AX \rtimes BY = (B \otimes A)(X \rtimes Y)$$
$$RA \rtimes SB = (R \rtimes S)(A \otimes B).$$

Proof. Using Remark 1 and Proposition 3, we have

$$AX \rtimes BY = BY \otimes AX = (B \otimes A)(Y \otimes X) = (B \otimes A)(X \rtimes Y),$$
$$RA \rtimes SB = RA \otimes SB = (R \rtimes S)(A \otimes B) = (R \rtimes S)(A \otimes B).$$

□

Theorem 1. *The right semi-tensor product is associative.*

Proof. We consider four cases as follows.
Case 1: $A \succ B \succ C$, let $A \in \mathscr{M}_{r,sp}(\mathfrak{S})$, $B \in \mathscr{M}_{p,qm}(\mathfrak{S})$ and $C \in \mathscr{M}_{m,n}(\mathfrak{S})$.

Using Lemma 1, we obtain

$$\begin{aligned}(A \rtimes B) \rtimes C &= [A(I_s \otimes B)] \rtimes C \\ &= A(I_s \otimes B)(I_{sq} \otimes C) \\ &= A(I_s \otimes B)(I_s \otimes I_q \otimes C) \\ &= A[(I_s I_s) \otimes B(I_q \otimes C)] \\ &= A[I_s \otimes (B \rtimes C)] \\ &= A \rtimes (B \rtimes C).\end{aligned}$$

Case 2: $A \prec B \prec C$, let $A \in \mathscr{M}_{m,n}(\mathbb{S})$, $B \in \mathscr{M}_{nt,q}(\mathbb{S})$ and $C \in \mathscr{M}_{qr,s}(\mathbb{S})$. We have by Lemma 1 that

$$\begin{aligned}A \rtimes (B \rtimes C) &= A \rtimes (I_r \otimes B)C \\ &= (I_{rt} \otimes A)(I_r \otimes B)C \\ &= (I_r \otimes I_t \otimes A)(I_r \otimes B)C \\ &= [I_r I_r \otimes (I_t \otimes A)B]C \\ &= [I_r \otimes (A \rtimes B)]C \\ &= (A \rtimes B) \rtimes C.\end{aligned}$$

Case 3: $A \prec B$ and $B \succ C$. For any $A \in \mathscr{M}_{m,n}(\mathbb{S})$, $B \in \mathscr{M}_{nt,pr}(\mathbb{S})$ and $C \in \mathscr{M}_{r,s}(\mathbb{S})$, we have

$$\begin{aligned}A \rtimes (B \rtimes C) &= A \rtimes B(I_p \otimes C) \\ &= (I_t \otimes A)B(I_p \otimes C) \\ &= (A \rtimes B)(I_p \otimes C) \\ &= (A \rtimes B) \rtimes C.\end{aligned}$$

Case 4: $A \succ B$ and $B \prec C$, consider $A \in \mathscr{M}_{m,pq}(\mathbb{S})$, $B \in \mathscr{M}_{p,n}(\mathbb{S})$ and $C \in \mathscr{M}_{nr,s}(\mathbb{S})$. For a subcase $r = bq$ where $b \in \mathbb{N}$, we have

$$\begin{aligned}A \rtimes (B \rtimes C) &= A \rtimes (I_{bq} \otimes B)C \\ &= (I_b \otimes A)(I_{bq} \otimes B)C \\ &= (I_b \otimes A)(I_b \otimes I_q \otimes B)C \\ &= [(I_b I_b) \otimes A(I_q \otimes B)]C \\ &= [I_b \otimes (A \rtimes B)]C \\ &= (A \rtimes B) \rtimes C.\end{aligned}$$

For another subcase $q = ar$ where $a \in \mathbb{N}$, we have

$$\begin{aligned}(A \rtimes B) \rtimes C &= [A(I_{ar} \otimes B)] \rtimes C \\ &= A(I_{ar} \otimes B)(I_a \otimes C) \\ &= A(I_a \otimes I_r \otimes B)(I_a \otimes C) \\ &= A[I_a I_a \otimes (I_r \otimes B)C] \\ &= A[I_a \otimes (B \rtimes C)]\end{aligned}$$

$$= A \rtimes (B \rtimes C).$$

In all cases, we conclude that the right semi-tensor product is associative. □

Theorem 2. *The map $(A,B) \mapsto A \rtimes B$ is bilinear. More precisely, for any matrices A, B, C over $\mathbb{S}$ with appropriate sizes and scalars $\alpha, \beta \in \mathbb{S}$, the following relations hold:*

$$(\alpha A + \beta B) \rtimes C = \alpha(A \rtimes C) + \beta(B \rtimes C), \tag{3}$$

$$C \rtimes (\alpha A + \beta B) = \alpha(C \rtimes A) + \beta(C \rtimes B). \tag{4}$$

Proof. We shall prove only the relation (4) since another one can be proved in a similar manner. Let us consider the case $C \prec A$, and let $C \in \mathscr{M}_{p,m}(\mathbb{S})$ and $A,B \in \mathscr{M}_{mt,n}(\mathbb{S})$. We have by Lemma 1 that

$$\begin{aligned} C \rtimes (\alpha A + \beta B) &= (I_t \otimes C)(\alpha A + \beta B) \\ &= \alpha(I_t \otimes C)A + \beta(I_t \otimes C)B \\ &= \alpha(C \rtimes A) + \beta(C \rtimes B). \end{aligned}$$

For the case $C \succ A$, let $C \in \mathscr{M}_{p,mt}(\mathbb{S})$ and $A,B \in \mathscr{M}_{m,n}(\mathbb{S})$. Again, Lemma 1 yields

$$\begin{aligned} C \rtimes (\alpha A + \beta B) &= C[I_t \otimes (\alpha A + \beta B)] \\ &= C[\alpha(I_t \otimes A) + \beta(I_t \otimes B)] \\ &= \alpha C(I_t \otimes A) + \beta C(I_t \otimes B) \\ &= \alpha(C \rtimes A) + \beta(C \rtimes B). \end{aligned}$$

□

Theorem 3. *If $A \rtimes B$ is well-defined, then $(A \rtimes B)^T = B^T \rtimes A^T$.*

Proof. To consider the case $A \succ B$, let $A \in \mathscr{M}_{m,nt}(\mathbb{S})$ and $B \in \mathscr{M}_{n,p}(\mathbb{S})$. By Lemma 1, we have

$$(A \rtimes B)^T = [A(I_t \otimes B)]^T = (I_t \otimes B)^T A^T = (I_t \otimes B^T)A^T = B^T \rtimes A^T.$$

For the case $A \prec B$, let $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{nt,p}(\mathbb{S})$. We have

$$(A \rtimes B)^T = [(I_t \otimes A)B]^T = B^T(I_t \otimes A)^T = B^T(I_t \otimes A^T) = B^T \rtimes A^T,$$

again we apply Lemma 1. □

The right semi-tensor product satisfies the following identity-like properties:

Theorem 4.
1. *If $A \in \mathscr{M}_{m,pn}(\mathbb{S})$, then $A \rtimes I_n = A$.*
2. *If $A \in \mathscr{M}_{m,p}(\mathbb{S})$, then $A \rtimes I_{pn} = I_n \otimes A$.*
3. *If $A \in \mathscr{M}_{pm,n}(\mathbb{S})$, then $I_p \rtimes A = A$.*
4. *If $A \in \mathscr{M}_{m,n}(\mathbb{S})$, then $I_{pm} \rtimes A = I_p \otimes A$.*

Proof.
1. $A \rtimes I_n = A(I_p \otimes I_n) = AI_{pn} = A$.
2. $A \rtimes I_{pn} = (I_n \otimes A)I_{pn} = I_n \otimes A$.
3. $I_p \rtimes A = (I_m \otimes I_p)A = I_{mp}A = A$.
4. $I_{pm} \rtimes A = I_{pm}(I_p \rtimes A) = (I_p \otimes I_m)(I_p \otimes A) = I_pI_p \otimes I_mA = I_p \otimes A$. □

Theorem 5. *Let A and B be square matrices such that the product $A \rtimes B$ is well-defined. If A or B is invertible, then $A \rtimes B \sim B \rtimes A$.*

Proof. We may assume that A is invertible. By Lemma 1, $I_t \otimes A$ is also invertible. For the case $A \succ_t B$, we have

$$\begin{aligned} A^{-1}(A \rtimes B)A &= A^{-1}A(I_t \otimes B)A \\ &= (I_t \otimes B)A \\ &= B \rtimes A. \end{aligned}$$

For the case $A \prec_t B$, we have by Lemma 1 that

$$\begin{aligned} (I_t \otimes A)^{-1}(A \rtimes B)(I_t \rtimes A) &= (I_t \otimes A^{-1})(I_t \otimes A)B(I_t \otimes A) \\ &= (I_t I_t \otimes A^{-1}A)B(I_t \otimes A) \\ &= B(I_t \otimes A) \\ &= B \rtimes A. \end{aligned}$$

From both cases, we can conclude that $A \rtimes B \sim B \rtimes A$. □

Theorem 6. *Let A and B be square matrices such that the product $A \rtimes B$ is well-defined. If both A and B are invertible, then $A \rtimes B$ is invertible and $(A \rtimes B)^{-1} = B^{-1} \rtimes A^{-1}$.*

Proof. For the case $A \succ_t B$, we have $A^{-1} \succ_t B^{-1}$ and thus by Lemma 1

$$\begin{aligned} (A \rtimes B)(B^{-1} \rtimes A^{-1}) &= A(I_t \otimes B)(I_t \otimes B^{-1})A^{-1} \\ &= A(I_t I_t \otimes BB^{-1})A^{-1} \\ &= AA^{-1} \\ &= I. \end{aligned}$$

For another case $A \prec_t B$, we have $A^{-1} \prec_t B^{-1}$ and thus by Lemma 1

$$\begin{aligned} (A \rtimes B)(B^{-1} \rtimes A^{-1}) &= (I_t \otimes A)BB^{-1}(I_t \otimes A^{-1}) = (I_t \otimes A)(I_t \otimes A^{-1}) \\ &= I_t I_t \otimes AA^{-1} = I_{nt}. \end{aligned}$$

From both cases, we conclude that $A \rtimes B$ is invertible and $(A \rtimes B)^{-1} = B^{-1} \rtimes A^{-1}$. □

Theorem 7. *Let A and B be square matrices such that the product $A \rtimes B$ is well-defined. If both A and B are orthogonal, then so is $A \rtimes B$.*

Proof. First, let us consider the case that $A \in M_{nt,nt}(\mathfrak{S})$ and $B \in M_{n,n}(\mathfrak{S})$. Assume $A^T A = I_{nt}$ and $B^T B = I_n$. Then $A \rtimes B = A(I_t \otimes B)$ and thus by Lemma 1

$$\begin{aligned} (A \rtimes B)^T(A \rtimes B) &= [A(I_t \otimes B)]^T[A(I_t \otimes B)] \\ &= (I_t \otimes B)^T A^T A(I_t \otimes B) \\ &= (I_t \otimes B^T)(I_t \otimes B) \\ &= I_t \otimes B^T B \\ &= I_{nt}. \end{aligned}$$

Now, consider $A \in M_{n,n}(\mathfrak{S})$ and $B \in M_{nt,nt}(\mathfrak{S})$ such that $A^TA = I_n$ and $B^TB = I_{nt}$. We have $A \rtimes B = (I_t \otimes A)B$ and then Lemma 1 yields

$$\begin{aligned}(A \rtimes B)^T(A \rtimes B) &= [(I_t \otimes A)B]^T[(I_t \otimes A)B] \\ &= B^T(I_t \otimes A^T)(I_t \otimes A)B \\ &= B^T[(I_tI_t \otimes A^TA]B \\ &= B^T(I_t \otimes I_n)B \\ &= B^TB \\ &= I_{nt}.\end{aligned}$$

Hence, $A \rtimes B$ is orthogonal. □

Proposition 5. *Let A and B be square matrices such that the product $A \rtimes B$ is well-defined. If both A and B are upper triangular, then so is $A \rtimes B$. Similar statements hold for the lower-triangular case and the diagonal case.*

Proof. We may assume that $A \in M_{nt,nt}(\mathfrak{S})$ and $B \in M_{n,n}(\mathfrak{S})$. Then $A \rtimes B = A(I_t \otimes B)$. Since B is upper triangular, so is $I_t \otimes B$. Thus, as being the usual product between upper triangular matrices, the matrix $A \rtimes B$ is also upper triangular. The statements for lower triangular matrices and diagonal matrices can be similarly proven. □

Although $A \rtimes B \neq B \rtimes A$ in general, their traces always coincide:

Proposition 6. *If $A \rtimes B$ and $B \rtimes A$ are square well-defined matrices, then* $\operatorname{tr}(A \rtimes B) = \operatorname{tr}(B \rtimes A)$.

Proof. Recall that $\operatorname{tr}(PQ) = \operatorname{tr}(QP)$ for any square matrices P, Q over $\mathfrak{S}$. For the case $A \prec_t B$, we have

$$\operatorname{tr}(A \rtimes B) = \operatorname{tr}(I_t \otimes A)B = \operatorname{tr}B(I_t \otimes A) = \operatorname{tr}(B \rtimes A).$$

The case $A \succ_t B$ can be treated in a similar way. □

4. The Right Semi-Tensor Product, Commutation Matrices and Vectorizations

In this section, relations between the right semi-tensor product, the left semi-tensor product, and certain kinds of vectorizations are discussed.

Let us recall some fundamental facts about commutation matrices and two kinds of vectorizations, namely, row/column vector operators. The *row vector operator* V_r is defined for each $A \in \mathscr{M}_{m,n}(\mathfrak{S})$ by

$$\mathrm{V}_r(A) = \begin{bmatrix}\mathrm{Row}_1(A) & \cdots & \mathrm{Row}_m(A)\end{bmatrix}^T.$$

The *column vector operator* V_c is defined for each $A \in \mathscr{M}_{m,n}(\mathfrak{S})$ by

$$\mathrm{V}_c(A) = \begin{bmatrix}\mathrm{Col}_1(A) \\ \vdots \\ \mathrm{Col}_n(A)\end{bmatrix}.$$

Let $X = [x_{ij}] \in \mathscr{M}_{m,n}(\mathbb{S})$. Let $E_{ij} \in \mathscr{M}_{m,n}(\mathbb{S})$ be the matrix having 1 in (i,j)-position and all other entries are zero. We can write

$$X = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}E_{ij}$$

which implies that

$$X^T = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}E_{ij}^T = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}^T X E_{ij}^T.$$

It follows that

$$\mathrm{V}_c(X^T) = \left\{\sum_{i=1}^{m}\sum_{j=1}^{n}(E_{ij}\otimes E_{ij}^T)\right\}\mathrm{V}_c(X)$$

So, we define the commutation matrix or the swap matrix $W_{[m,n]}$ by

$$K_{[m,n]} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}\otimes E_{ij}^T \in \mathscr{M}_{mn,mn}(\mathbb{S}).$$

Thus, $K_{[m,n]}$ possess the property that $\mathrm{V}_c(X^T) = K_{[m,n]}\mathrm{V}_c(X)$ for all $X \in \mathscr{M}_{m,n}(\mathbb{S})$. Let us see two examples:

$$K_{[2,3]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K_{[3,2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that every commutation matrix is a permutation matrix. In particular, $K_{[m,n]}$ is invertible and $K_{[m,n]}^{-1} = K_{[n,m]} = K_{[n,m]}^T$.

Lemma 2 ([23]). *For any $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{p,q}(\mathbb{S})$, we have*

$$K_{[m,p]}(A\otimes B)K_{[n,q]}^T = B\otimes A.$$

The right semi-tensor product and the left semi-tensor product are related through commutation matrices as follows.

Theorem 8. *Let $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and $B \in \mathscr{M}_{p,q}(\mathbb{S})$. If $A \succ_t B$, then*

$$A \rtimes B = A \ltimes K_{[p,t]} \ltimes B \ltimes K_{[t,q]},$$

$$A \ltimes B = A \rtimes K_{[t,p]} \rtimes B \ltimes K_{[q,t]}.$$

If $A \prec_t B$, then

$$A \rtimes B = K_{[m,t]} \ltimes A \ltimes K_{[t,n]} \ltimes B,$$

$$A \ltimes B = K_{[t,m]} \rtimes A \ltimes K_{[n,t]} \rtimes B.$$

Proof. For the case $A \succ_t B$, using Lemma 2, we obtain

$$A \ltimes K_{[p,t]} \ltimes B \ltimes K_{[t,q]} = AK_{[p,t]}(B\otimes I_t)K_{[t,q]} = A(I_t\otimes B) = A \rtimes B.$$

On the other hand, we have by Lemma 2 that

$$\begin{aligned} A \rtimes K_{[t,p]} \rtimes B \ltimes K_{[q,t]} &= A \rtimes K_{[t,p]} \rtimes \{(I_t \otimes B)K_{[q,t]}\} \\ &= A \rtimes K_{[t,p]} \rtimes \{K_{[p,t]}(B \otimes I_t)\} \\ &= AK_{[t,p]}K_{[p,t]}(B \otimes I_t) \\ &= A(B \otimes I_t) \\ &= A \ltimes B. \end{aligned}$$

The case $A \prec_t B$ can be proved in a similar way. □

In particular for row/column vectors, we have the following relations.

Corollary 1. *For any $R \in \mathscr{M}_{1,np}(\mathfrak{S})$ and $C \in \mathscr{M}_{p,1}(\mathfrak{S})$, we have*

$$R \rtimes C = (RK_{[p,n]}) \ltimes C,$$

$$R \ltimes C = (RK_{[n,p]}) \rtimes C.$$

For any $R \in \mathscr{M}_{1,p}(\mathfrak{S})$ and $C \in \mathscr{M}_{np,1}(\mathfrak{S})$, we have

$$R \rtimes C = R \ltimes (K_{[n,p]}C),$$

$$R \ltimes C = R \ltimes (K_{[p,n]}C).$$

Next, we discuss relations between vector operators and the right semi-tensor product.

Lemma 3 ([23])**.** *For any $A \in \mathscr{M}_{m,n}(\mathfrak{S})$, $B \in \mathscr{M}_{n,q}(\mathfrak{S})$ and $C \in \mathscr{M}_{p,m}(\mathfrak{S})$, we have*

$$\mathrm{V}_c(ABC) = (C^T \otimes A)\mathrm{V}_c(B).$$

Proposition 7. *For any $A \in \mathscr{M}_{m,n}(\mathfrak{S})$, $B \in \mathscr{M}_{n,q}(\mathfrak{S})$ and $C \in \mathscr{M}_{p,m}(\mathfrak{S})$, we have*

$$\mathrm{V}_r(CA) = A^T \rtimes \mathrm{V}_r(C),$$

$$\mathrm{V}_c(AB) = A \rtimes \mathrm{V}_c(B).$$

Proof. Note that $\mathrm{V}_r(M) = \mathrm{V}_c(M^T)$ for any matrix M over $\mathfrak{S}$. It follows from Lemma 3 that

$$\mathrm{V}_r(CA) = \mathrm{V}_c(A^TC^T) = (I_p \otimes A^T)\mathrm{V}_c(C^T) = (I_p \otimes A^T)\mathrm{V}_r(C) = A^T \rtimes \mathrm{V}_r(C),$$

$$\mathrm{V}_c(AB) = (I_q \otimes A)\mathrm{V}_c(B) = A \rtimes \mathrm{V}_c(B).$$

□

5. Conclusion

We extend the notion of right semi-tensor product for real matrices to that for matrices over an arbitrary commutative semiring. The right semi-tensor product is defined for any pair of matrices satisfying the matching-dimension condition, and is given in terms of the tensor product. The right semi-tensor product includes the usual matrix product and the scalar multiplication as special cases. The right semi-tensor product turns out to be an associative bilinear map that is compatible with the transposition and the inversion. The product also satisfies certain identity-like properties. The product preserves some structural properties of matrices: invertible, orthogonal, upper/lower triangular, and diagonal. We can convert between

the right semi-tensor product of two matrices to the left semi-tensor product using commutation matrices. Moreover, the row/column vector operator of the usual product of matrices can be written in terms of the right semi-tensor product.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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