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Research Article

Numerical Integration of Singularly Perturbed Differential-Difference Problem Using Non Polynomial Interpolating Function

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Abstract. In this paper, a simple integration of differential-difference problem with singular perturbed nature using non polynomial interpolating function is presented. Firstly, an equivalent first-order problem of the given second order singularly perturbed equation. Resulting first order problem is solved by numerical integration using the non polynomial interpolating function. To analyse the method computationally, several model experiments have been solved and results are compared with upwind method for different values for the advanced, delay and the perturbation parameter. The cause of the small parameters on the layer solutions are presented in graphs.

Keywords. Singularly perturbed differential-difference problem; Layer behaviour; Numerical integration

MSC. 65L10; 65L11; 65L12

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1. Introduction

Functional Differential Equations are also called as singularly perturbed differential-difference Equations, are mathematical models for the real life phenomenon. Their applications exist in all areas of contemporary sciences such as economics, engineering, evolutionary biology, biomechanics and physics [1, 16]. The bistable devices [4, 15], the first exit time problem in the modeling of the activation of neuronal variability [19, 22] are all examples involving equations of differential-difference type. Lange and Miura [10–14] have carried out a detailed study on the solutions of SPDDEs, turning point behaviour, resonance behaviour and boundary and interior layer behaviour. Kadalbajoo and Sharma [6] gives a method with finite difference operator of upwind on a mesh of special type. Kadalbajoo and Sharma [7] analysed a method to solve singularly perturbed problem with mixed shifts. Swamy *et al*. [20] presented Galerkin method with fitting to solve such equations having layer behaviour with delay and advanced shifts. Salama and Al-Amery [18] has given a asymptotic numerical numerical method for solving boundary value problem with shift arguments.

Chakravarthy *et al*. [3] presented a fitted finite difference method for second order delay differential equation with large delay using cubic spline in compression. Kanth and Murali [9] presented a scheme for a convection delayed dominated diffusion problem using tension splines.

Chakravarthy and Rao [2] proposed a modified Numerov method for differential-difference equations of singularly perturbed with mixed shifts. In [17], Kanth and Murali has given a numerical method based on parametric cubic spline for a nonlinear singularly perturbed delay differential problem.

In this paper, a simple numerical integration of singularly perturbed differential-difference equations using non polynomial interpolating function is presented. In Section 2, we described the numerical scheme for left-end and right-end layer problems. Section 3 deals with the numerical results of the model problems. Finally discussions and conclusion were given in final section.

2. Numerical Scheme

We consider the following singularly perturbed differential-difference equation

$$\varepsilon y''(t) + a(t)y'(t) + b(t)y(t-\delta) + d(t)y(t) + c(t)y(t+\eta) = h(t) \tag{1}$$

$\forall\ t \in (0,1)$ with the interval boundary conditions

$$y(t) = \phi(t), \quad \text{on} \quad -\delta \le t \le 0, \tag{2}$$

$$y(t) = \gamma(t), \quad \text{on} \quad 1 \le t \le 1+\eta, \tag{3}$$

where a, b, c, d, h, ϕ and γ are bounded and functions of continuously differentiable of t on the given domain, $0 < \varepsilon \ll 1$ taken as the perturbation parameter and $0 < \delta \ll 1$ and $0 < \eta \ll 1$ are taken as the delay and the advance parameters. With the help of Taylor series in the vicinity of x, we have

$$y(t-\delta) \approx y(t) - \delta y'(t), \tag{4}$$

$$y(t+\eta) \approx y(t) + \eta y'(t). \tag{5}$$

Using (4) and (5) in (1), we get an equivalent singularly perturbed boundary value problem:

$$\varepsilon y''(t) + p(t)y'(t) + Q(t)y(t) = h(t) \tag{6}$$

with

$$y(0) = \phi(0) = \phi, \tag{7}$$

$$y(1) = \gamma(1) = \gamma, \tag{8}$$

where

$$p(t) = a(t) + c(t)\eta - b(t)\delta \tag{9}$$

and

$$Q(t) = b(t) + c(t) + d(t). \tag{10}$$

This transformation from (1) to (6) is permitted, because of the condition that $0 < \delta \ll 1$ and $0 < \eta \ll 1$ are sufficiently small [5]. Therefore, the solution of (6) will give a relative solution to (1).

2.1 Left-End Boundary Layer Problems

Taking $Q(t) = b(t) + c(t) + d(t) \leq 0, p(t) = a(t) + c(t)\eta - b(t)\delta \geq L > 0$ over the domain $[0,1]$, L is positive constant then (6) possess a unique solution and $y(t)$ shows a layer on the left-end of the domain, i.e., at $t = 0$.
Rewrite (6) as follows:

$$\varepsilon y''(t) + (p(t)y(t))' + q(t)y(t) = h(t) \tag{11}$$

where $q(t) = Q(t) - p'(t)$, with

$$y(0) = \phi(0) = \phi \;\text{ and }\; y(1) = \gamma(1) = \gamma. \tag{12}$$

The numerical scheme is explained by the following steps:

Step 1. By setting $\varepsilon = 0$ in (11), get the reduced problem and solve it with the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem, that is,

$$(p(t)y_0(t))' + q(t)y_0(t) = h(t) \tag{13}$$

with

$$y_0(1) = \gamma. \tag{14}$$

Step 2. The approximate expression to the (11) is taken as:

$$\varepsilon y''(t) + (p(t)y(t))' + q(t)y_0(t) = h(t) \tag{15}$$

with

$$y(0) = \phi(0) = \phi \;\text{ and }\; y(1) = \gamma(1) = \gamma. \tag{16}$$

Step 3. The approximated second order problem (15)-(16) is replaced by relative first order problem as follows.:
By integrating (15), we obtain:

$$\varepsilon y'(t) + p(t)y(t) = f(t) + K \tag{17}$$

where

$$f(t) = \int H(t)dt \;\text{ and }\; H(t) = h(t) - q(t)y_0(t) \tag{18}$$

and K is an integrating constant to be determined.
To determine the value of K, take the condition that (17) should satisfy the condition, $y(1) = \gamma$ i.e., $p(1)y(1) = f(1) + K$.
Therefore, $K = p(1)\gamma - f(1)$.

Therefore, the original problem (11)-(12) is transformed to an equivalent first order problem (17) with $y(0) = \phi$, which is in turn a relative approximation to (1)-(3).
To solve this initial value problem, we have used a new numerical scheme described below:
Assume that the solution $y(t)$ to the problem (17) can be represented in the interval $[t_n, t_{n+1}]$, $n \geq 0$ as the non polynomial interpolating function:

$$Y(t) = (\alpha_1 + \alpha_2)e^{-2t} + \alpha_3 t^2 + \alpha_4 t + \alpha_5, \tag{19}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are real undetermined coefficients α_5 is constant.
Let y_n is the numerical estimate to the theoretical solution $y(t)$ at t_n and $f_n = f(t_n, y_n)$.
We define mesh points as

$$t_n = a + nh, \quad n = 0, 1, 2, \dots . \tag{20}$$

Imposing the following constraints on the interpolating function (19) in order to get the undetermined coefficients. The interpolating function must coincide with the theoretical solution at $t = t_n$ and $t = t_{n+1}$. Hence, we have

$$Y(t_n) = (\alpha_1 + \alpha_2)e^{-2t_n} + \alpha_3 t_n^2 + \alpha_4 t_n + \alpha_5, \tag{21}$$

$$Y(t_{n+1}) = (\alpha_1 + \alpha_2)e^{-2t_{n+1}} + \alpha_3 t_{n+1}^2 + \alpha_4 t_{n+1} + \alpha_5. \tag{22}$$

Further, the derivatives of the interpolating function are required to coincide with the first,second, and third derivatives with respect to t at $t = t_n$. Denote the ith total derivative of $f(t, y)$ with respect to t with $f^{(i)}$ such that

$$Y'(t_n) = f_n, \tag{23}$$

$$Y''(t_n) = f_n^1, \tag{24}$$

$$Y'''(t_n) = f_n^2. \tag{25}$$

The derivatives of the interpolant are

$$f_n = -2(\alpha_1 + \alpha_2)e^{-2t_n} + 2\alpha_3 t_n^2 + \alpha_4, \tag{26}$$

$$f_n^1 = 4(\alpha_1 + \alpha_2)e^{-2t_n} + 2\alpha_3, \tag{27}$$

$$f_n^2 = -8(\alpha_1 + \alpha_2)e^{-2t_n}. \tag{28}$$

Using (28), we have

$$\alpha_1 + \alpha_2 = -\frac{1}{8} f_n^2 e^{2t_n}. \tag{29}$$

Substituting (29) into (27), we get

$$\alpha_3 = \frac{1}{2}\left(f_n^1 + \frac{1}{2} f_n^2\right). \tag{30}$$

Using (29) and (30) into (26), we get

$$\alpha_4 = (f_n - \frac{1}{4} f_n^2) - (f_n^1 + \frac{1}{2} f_n^2) t_n. \tag{31}$$

Since $Y(t_{n+1}) = y_{n+1}$ and $Y(t_n) = y_n$ implies that $y(t_{n+1}) = y_{n+1}$ and $y(t_n) = y_n$

$$Y(t_{n+1}) - Y(t_n) = y_{n+1} - y_n.$$

Then, we have

$$y_{n+1} - y_n = (\alpha_1 + \alpha_2)\left[e^{-2t_{n+1}} - e^{-2t_n}\right] + \alpha_3\left[t_{n+1}^2 - t_n^2\right] + \alpha_4\left[t_{n+1} - t_n\right]. \tag{32}$$

The nodal points are

$$t_n = a + nh, t_{n+1} = a + (n+1)h \text{ with } n = 0,1,2,\dots \tag{33}$$

Substitute (29), (30) and (31) into (32), we get

$$y_{n+1} = y_n - \frac{1}{8}f_n^2(e^{-2h} - 1) + \frac{1}{2}\left(f_n^1 + \frac{1}{2}f_n^2\right)h^2 + \left(f_n - \frac{1}{4}f_n^2\right)h. \tag{34}$$

Hence, (34) is the new scheme for solution of the first order differential equation (17).

2.2 Right-End Boundary Layer Problems

Now, the scheme is extended to the right-end layer problems on the underlying domain. Consider (6) rewritten as follows, for convenience:

$$\varepsilon y''(t) + (p(t)y(t))' + q(t)y(t) = h(t) \tag{35}$$

with

$$y(0) = \phi(0) = \phi \text{ and } y(1) = \gamma(1) = \gamma, \tag{36}$$

where $p(t) = a(t) + c(t)\eta - b(t)\delta$, $q(t) = Q(t) - p'(t)$ and $Q(t) = b(t) + c(t) + d(t)$. Further, assume that $p(t) = a(t) + c(t)\eta - b(t)\delta \le R < 0$ over the interval $[0,1]$, where R is a negative constant. With this assumption, the layer will be in the vicinity of $t = 1$.

Step 1. Let the solution of the reduced problem be $y_0(x)$ of (22), that is,

$$(p(t)y_0(t))' + q(t)y_0(t) = h(t) \tag{37}$$

with

$$y_0(0) = \phi. \tag{38}$$

Step 2. The approximate equation to the (22) is

$$\varepsilon y''(t) + (p(t)y(t))' + q(t)y_0(t) = h(t) \tag{39}$$

with $y(0) = \phi(0) = \phi$ and $y(1) = \gamma(1) = \gamma$, where the term $y(x)$ is replaced by the solution $y_0(t)$ of the reduced problem (37)-(38).

Step 3. The approximated problem (39) is transformed to an asymptotically equivalent first order problem as follows.

By integrating (39), we get

$$\varepsilon y'(t) + p(t)y(t) = f(t) + K, \tag{40}$$

where

$$f(t) = \int H(t)dt \text{ and } H(t) = h(t) - q(t)y_0(t) \tag{41}$$

and K is an integrating constant to be determined. To find the value of the constant K, use the condition that the reduced equation of (40) should satisfy the condition, $y(0) = \phi$ i.e., $p(0)y(0) = f(0) + K$. Therefore, $K = p(0)\phi - f(0)$. In this case, we have used a new numerical scheme described below:

Assume that the solution $y(x)$ to the problem (40) can be represented in the interval $[t_{n-1}, t_n]$, $n \ge 1$ as the non polynomial interpolating function:

$$F(t) = (\alpha_1 + \alpha_2)e^{-2t} + \alpha_3 t^2 + \alpha_4 t + \alpha_5, \tag{42}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are real undetermined coefficients α_5 is constant.

We assume y_n is a numerical estimate to the theoretical solution $y(t)$ at t_n and $f_n = f(t_n, y_n)$.
We define mesh points as follows:

$$t_{n-1} = a + (n-1)h, \quad n = 1,2,3,\dots. \tag{43}$$

Imposing the following constraints on the interpolating function (42) in order to get the undetermined coefficients.
The interpolating function must coincide with the theoretical solution at $t = t_n$ and $t = t_{n-1}$.
Hence we require that

$$Y(t_n) = (\alpha_1 + \alpha_2)e^{-2t_n} + \alpha_3 t_n^2 + \alpha_4 t_n + \alpha_5, \tag{44}$$

$$Y(t_{n-1}) = (\alpha_1 + \alpha_2)e^{-2t_{n-1}} + \alpha_3 t_{n-1}^2 + \alpha_4 t_{n-1} + \alpha_5. \tag{45}$$

Further the derivatives of the interpolating function are required to coincide with the first, second, and third derivatives with respect to t at $t = t_n$. Denote the ith total derivative of $f(t,y)$ with respect to t with $f^{(i)}$ such that

$$Y'(t_n) = f_n, \tag{46}$$

$$Y''(t_n) = f_n^1, \tag{47}$$

$$Y'''(t_n) = f_n^2. \tag{48}$$

The derivatives of the interpolant are

$$f_n = -2(\alpha_1 + \alpha_2)e^{-2t_n} + 2\alpha_3 t_n^2 + \alpha_4, \tag{49}$$

$$f_n^1 = 4(\alpha_1 + \alpha_2)e^{-2t_n} + 2\alpha_3, \tag{50}$$

$$f_n^2 = -8(\alpha_1 + \alpha_2)e^{-2t_n}. \tag{51}$$

Solving for $\alpha_1 + \alpha_2$ from (51), we have

$$\alpha_1 + \alpha_2 = -\frac{1}{8} f_n^2 e^{2t_n}. \tag{52}$$

Substituting (52) into (50), we get

$$\alpha_3 = \frac{1}{2}\left(f_n^1 + \frac{1}{2} f_n^2\right). \tag{53}$$

Using (52) and (53) into (49), we have

$$\alpha_4 = \left(f_n - \frac{1}{4} f_n^2\right) - \left(f_n^1 + \frac{1}{2} f_n^2\right) t_n. \tag{54}$$

Since $Y(t_{n-1}) = y_{n-1}$ and $Y(t_n) = y_n$ implies that $y(t_{n-1}) = y_{n-1}$ and $y(t_n) = y_n$

$$Y(t_n) - Y(t_{n-1}) = y_n - y_{n-1}. \tag{55}$$

Then, we have

$$y_n - y_{n-1} = (\alpha_1 + \alpha_2)\left[e^{-2t_n} - e^{-2tn}\right] + \alpha_3\left[t_n^2 - t_{n-1}^2\right] + \alpha_4[t_n - t_{n-1}]. \tag{56}$$

Substitute (52), (53) and (54) into (56), we have

$$y_{n-1} = y_n + \frac{1}{8} f_n^2 (1 - e^{2h}) - \frac{1}{2}\left(f_n^1 + \frac{1}{2} f_n^2\right) h^2 + \left(f_n - \frac{1}{4} f_n^2\right) h. \tag{57}$$

Hence (57) is the scheme for solution of the first order differential equation (4).

3. Numerical Illustrations

Example 1. Numerical results for model problem (1)-(3) having boundary layer at the left-end with

$$a(t) = 1, \quad b(t) = 2, \quad c(t) = 0, \quad d(t) = -3, \quad h(t) = 0, \quad \phi(t) = 1, \quad \gamma(t) = 1$$

are shown in Table 1. The layer behaviour is shown graphically in Figure 1.

Table 1. With $\varepsilon = 10^{-3}$, maximum absolute errors of Example 1

δ/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	2.2030e-02	0.232912	9.9900e-04	0.012377	9.9900e-04	1.4359e-03
0.3ε	2.1988e-02	0.232753	1.0002e-03	0.012373	1.0002e-03	1.4358e-03
0.6ε	2.1945e-02	0.232594	1.0014e-03	0.012370	1.0014e-03	1.4356e-03
0.9ε	2.1903e-02	0.232436	1.0026e-03	0.012367	1.0026e-03	1.4355e-03

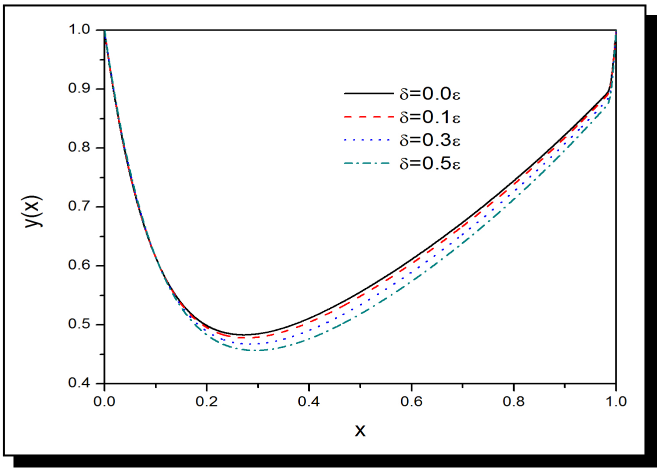

Figure 1. Layer behaviour in Example 1 for $\varepsilon = 0.1$

Example 2. Numerical results for Model problem (1)-(3) having boundary layer at the left end with

$$a(t) = 1, \quad b(t) = 0, \quad c(t) = 2, \quad d(t) = -3, \quad h(t) = 0, \quad \phi(t) = 1, \quad \gamma(t) = 1$$

are tabulated in Table 2. The layer behaviour is shown graphically in Figure 2.

Table 2. With $\varepsilon = 10^{-3}$, maximum absolute errors of Example 2

η/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	2.2030e-02	0.232912	9.9900e-04	0.012377	9.9900e-04	1.4359e-03
0.3ε	2.2073e-02	0.233071	9.9780e-04	0.012380	9.9780e-04	1.4361e-03
0.6ε	2.2115e-02	0.233229	9.9661e-04	0.012383	9.9661e-04	1.4362e-03
0.9ε	2.2158e-02	0.233388	9.9542e-04	0.012387	9.9542e-04	1.4364e-03

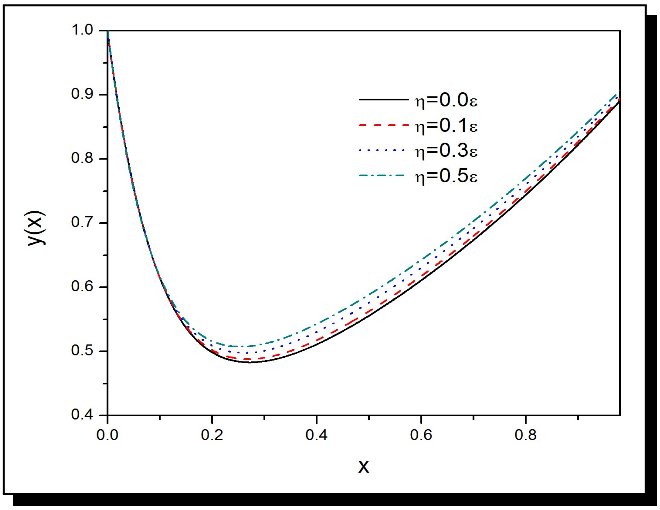

Figure 2. Layer behaviour in Example 2 for $\varepsilon = 0.1$

Example 3. Numerical results for Model problem given by equations (1)-(3) having boundary layer at the left end with

$$a(t) = 1, \quad b(t) = -2, \quad c(t) = 1, \quad d(t) = -5, \quad h(t) = 0, \quad \phi(t) = 1, \quad \gamma(t) = 1$$

are shown in Table 3 and 4. The layer behaviour is presented graphically in Figure 3 and 4.

Table 3. With $\eta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$, maximum absolute errors of Example 3

δ/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	3.2330e-02	0.365172	5.9583e-03	0.017050	5.9583e-03	5.9585e-03
0.3ε	3.2412e-02	0.365550	5.9512e-03	0.017064	5.9512e-03	5.9514e-03
0.6ε	3.2494e-02	0.365928	5.9441e-03	0.017078	5.9441e-03	5.9443e-03
0.9ε	3.2575e-02	0.366306	5.9370e-03	0.017092	5.9370e-03	5.9372e-03

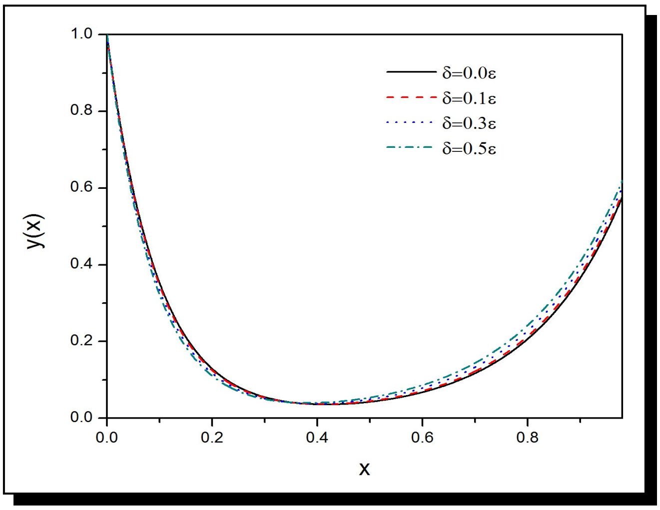

Figure 3. Layer behaviour in Example 3 for $\varepsilon = 0.1$, $\eta = 0.5$

Table 4. With $\delta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$, maximum absolute errors of Example 3

η/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	3.2398e-02	3.6549e-01	5.9524e-03	1.7062e-02	5.9524e-03	5.9526e-03
0.3ε	3.2439e-02	3.6568e-01	5.9488e-03	1.7069e-02	5.9488e-03	5.9490e-03
0.6ε	3.2480e-02	3.6587e-01	5.9453e-03	1.7076e-02	5.9453e-03	5.9455e-03
0.9ε	3.2521e-02	3.6605e-01	5.9417e-03	1.7083e-02	5.9417e-03	5.9419e-03

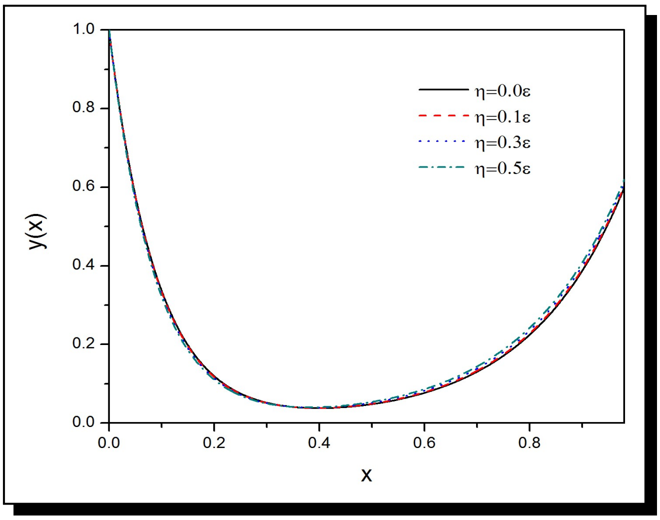

Figure 4. Layer behaviour in Example 3 for $\varepsilon = 0.1$, $\delta = 0.5\varepsilon$

Example 4. Numerical results for Model problem given by equations (1)-(3) having boundary layer at the right end with

$$a(t) = -1, \quad b(t) = -2, \quad c(t) = 0, \quad d(t) = 1, \quad h(t) = 0, \quad \phi(t) = 1, \quad \gamma(t) = -1$$

are tabulated in Table 5. Figure 5 shows the layer behaviour in the solution of the problem.

Table 5. With $\varepsilon = 10^{-3}$, maximum absolute errors of Example 4

δ/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	9.9900e-04	0.0009995	9.9900e-04	0.00099905	9.9900e-04	0.00099905
0.3ε	1.0002e-03	0.0010007	1.0002e-03	0.00100020	1.0002e-03	0.00100020
0.6ε	1.0014e-03	0.0010019	1.0014e-03	0.00100140	1.0014e-03	0.00100140
0.9ε	1.0026e-03	0.0010031	1.0026e-03	0.00100265	1.0026e-03	0.00100265

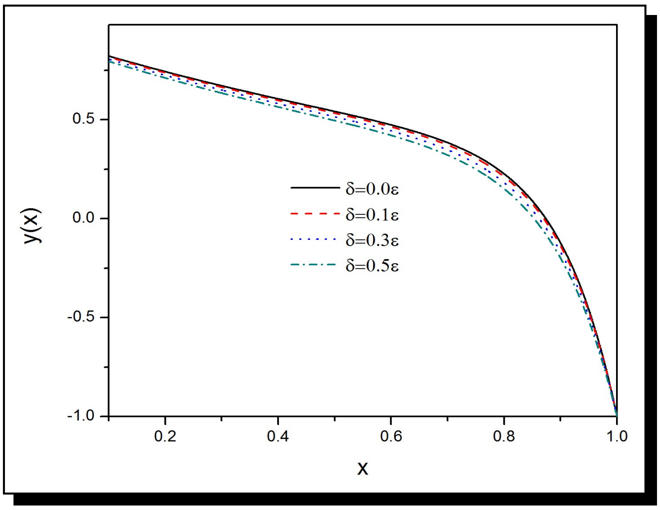

Figure 5. Layer behaviour in Example 4 for $\varepsilon = 0.1$

Example 5. Numerical results for Model problem given by equations (1)-(3) having boundary layer at the right end with

$$a(t) = -1, \quad b(t) = 0, \quad c(t) = -2, \quad d(t) = 1, \quad h(t) = 0, \quad \phi(t) = 1, \quad \gamma(t) = -1$$

are tabulated in Table 6. The layer behaviour in the solution is presented in Figure 6.

Table 6. With ε=10^{-3}, maximum absolute errors of Example 5

η/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.0099501	3.6322e+095	0.0009995	0.0009995	0.0001581	0.0009990
0.3ε	0.0099495	3.8832e+095	0.0009994	0.0009983	0.0001581	0.0009978
0.6ε	0.0099489	4.15113e+095	0.0009994	0.0009971	0.0001581	0.0009966
0.9ε	0.0099483	4.4377e+095	0.0009993	0.0009959	0.0001581	0.0009954

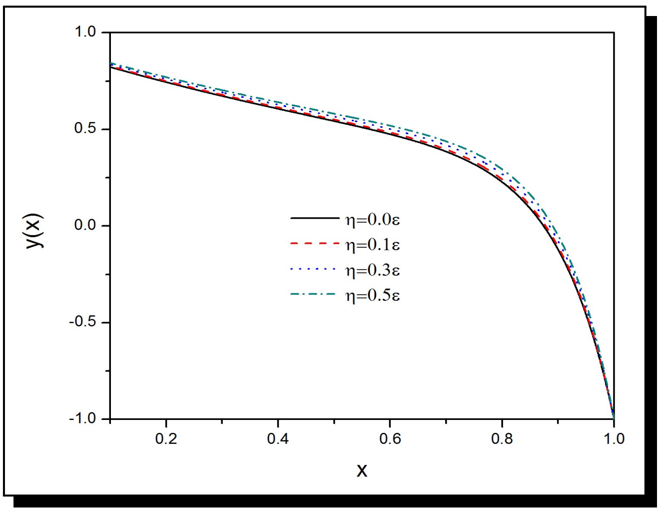

Figure 6. Layer behaviour in Example 5 for $\varepsilon = 0.1$, $\delta = 0.5\varepsilon$

Example 6. Numerical results for Model problem given by equations (1)-(3) having boundary layer at the right end with

$$a(t) = -1, \quad b(t) = -2, \quad c(t) = -2, \quad d(t) = 1, \quad h(t) = 0, \quad \phi(t) = 1, \quad \gamma(t) = -1$$

are shown in Table 7 and 8. The layer behaviour is shown graphically in Figure 7 and 8.

Table 7. Maximum absolute errors of Example 6 for $\delta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$

η/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	2.9970e-03	0.0030014	2.9970e-03	0.0029974	2.9970e-03	0.0029973
0.3ε	2.9934e-03	0.0029978	2.9934e-03	0.0029938	2.9934e-03	0.0029935
0.6ε	2.9898e-03	0.0029943	2.9898e-03	0.0029902	2.9898e-03	0.0029900
0.9ε	2.9863e-03	0.0029907	2.9863e-03	0.0029867	2.9863e-03	0.0029865

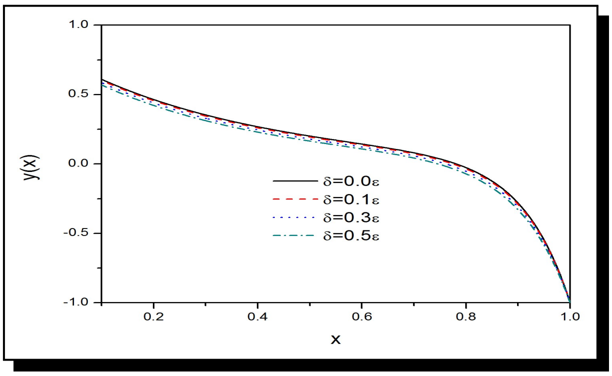

Figure 7. Layer behaviour in Example 6 for $\varepsilon = 0.1$, $\eta = 0.5\varepsilon$

Table 8. Maximum absolute errors of Example 6 for $\eta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$

δ/h	10^{-3}		10^{-4}		10^{-5}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	2.9851e-03	2.9895e-03	2.9851e-03	2.9855e-03	2.9851e-03	2.9852e-03
0.3ε	2.9886e-03	2.9931e-03	2.9886e-03	2.9891e-03	2.9886e-03	2.9887e-03
0.6ε	2.9922e-03	2.9966e-03	2.9922e-03	2.9927e-03	2.9992e-03	2.9923e-03
0.9ε	2.9958e-03	3.0002e-03	2.9958e-03	2.9962e-03	2.9958e-03	2.9959e-03

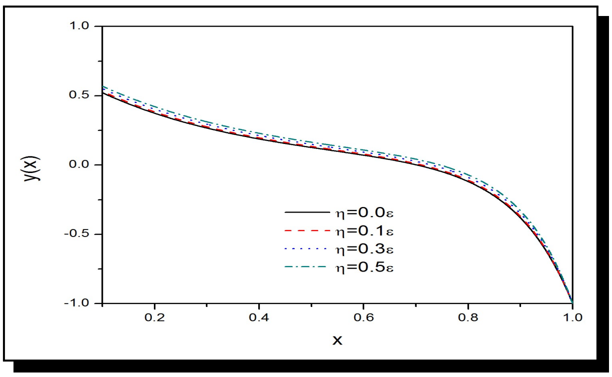

Figure 8. Layer behaviour in Example 6 for $\varepsilon = 0.1$, $\delta = 0.5\varepsilon$

4. Discussions and Conclusion

The numerical solution of a initial value problem is easier than that of corresponding boundary value problem. Hence, we desire to reduce the second order problem into a first order problem. This numerical integration scheme using non polynomial interpolation function provides an alternative to the formal approaches of converting the second order problems into first order problems. The method is implemented on several examples with left layer and right layer, with distinct values of the delay parameter δ, advanced parameter η and the perturbation ε. The computational results are tabulated with comparison. The affect of delay and advanced parameters on the solutions of the problem has been investigated through graphs. When the solution of the SPDDEs exhibit layer on the left-end, it is observed that the affect of delay or advanced parameters in the layer domain is negligible, while it is considerable in the outer region. The variation in the advanced parameter affects the solution in the similar manner as the change in delay affects but reversely in (see Figures 1-5). Also, there is an influence in layer region as well as outer region, when the SPDDEs exhibit layer on the right-end with respect to the variations in delay or advanced parameters. We noticed that, as the delay parameter increases the thickness of the layer increases, while the advanced parameter increases the layer thickness decreases (Figures 6-8). It can be noticed from results that the present scheme meets the exact solution very well.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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