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Research Article

The Generalized α-Nonexpansive Mappings and Related Convergence Theorems in Hyperbolic Spaces

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Abstract. In this paper, we propose and analyze a generalized α-nonexpansive mappings on a nonempty subset of a hyperbolic space i.e.,

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \alpha d(y,Tx) + \alpha d(x,Ty) + (1-2\alpha)d(x,y),$$

and prove Δ-convergence theorems and convergence theorems for a generalized α-nonexpansive mappings in a hyperbolic space.

Keywords. Fixed point set; Generalized α-nonexpansive mappings; Δ-convergence theorems and hyperbolic spaces

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1. Introduction

The existence of a fixed point is of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, finances, informatics, engineering and physics. Let M be a nonempty subset of a linear space X, let $F(T) = \{x \in M : Tx = x\}$ denotes the set of fixed points of the mapping T on M. Let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *nonexpansive*, if

$$d(Tx,Ty) \le d(x, y), \tag{1.1}$$

for each $x, y \in M$. Define a mapping T on $[0,1]$ by $Tx = \frac{x}{3}$, it's easy to see that T is nonexpansive. Let (X,d) be metric space and let M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *quasi-nonexpansive*, if

$$d(Tx, p) \le d(x, p)$$

for each $x \in M$ and $p \in F(T)$. Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & x \neq 3, \\ 2, & x = 3. \end{cases}$$

Then $F(T) = \{0\} \neq \emptyset$ and T is quasi-nonexpansive (see [20]). In the last sixty-five years, the numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings.

In 1953, Mann [13] has introduced The Mann iteration process is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is real sequences in $(0,1)$.

In 1974, Ishikawa [6] has introduced The Ishikawa iteration process is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$ and $\{y_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \ \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

In 2007, Agarwal *et al.* [2] introduced a new iteration process whose rate of convergence is similar to Picard iteration and faster than other fixed point iteration processes as follows: For M be a convex subset of a linear space X and $T : M \to M$ a mapping. Then the modified

S-iteration process is a sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = S(x_n, \alpha_n, \beta_n, T^n), \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

In 2008, Suzuki [20] introduced a class of single valued mappings called Suzuki-generalized nonexpansive mappings (or condition C). The condition C is weaker than nonexpansiveness and stronger than quasi-nonexpansive, as follows: Let T be a self-mapping on a subset M of a metric space X. Then T is said to satisfy *condition C* if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le d(x,y),$$

for each $x, y \in M$.

It is obvious that every nonexpansive mapping satisfies condition C, but the converse is not true, that is condition C is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. The next simple example can show this fact. We see that, if define a mapping $T : [0,3] \to [0,3]$ by

$$Tx = \begin{cases} 0, & x \ne 3, \\ \frac{2}{3}, & x = 3. \end{cases}$$

Then T is condition C, but T is not nonexpansive (see [20]).

In 2011, Aoyama and Kohsaka [3] introduced the class of α-nonexpansive mappings in Banach spaces. This class contains the class of nonexpansive mappings and is related to the class of firmly nonexpansive mappings in Banach spaces as follows: let X be a Banach space and M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *α-nonexpansive* for some real number $\alpha < 1$, if

$$\|Tx - Ty\| \le \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|, \tag{1.5}$$

for all $x, y \in C$. Clearly, 0-nonexpansive maps is exactly nonexpansive maps. The next simple example can show this fact. We see that, let $M = [0,4]$ is a subste of $\mathbb{R}$ endowed with the usual normand usual order. Define $T : M \to M$ by

$$Tx = \begin{cases} 0; & x \ne 4, \\ 2; & x = 4. \end{cases}$$

Then, T is a α-nonexpansive mapping with $\alpha \ge \frac{1}{2}$ (see [17]).

In 2011, Sahu [15] has introduced Normal S-iteration Process is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$ and $\{y_n\}$ in M is defined by sequence $\{x_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is real sequences in $(0,1)$.

In 2014, Kadioglu [7] defined Picard normal S-iteration process (PNS) is defined as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ is real sequences in $(0,1)$. If $\beta_n = 0$ and $\alpha_n = \beta_n = 0$ in (1.7) then it reduces to Normal S-iteration process and Picard iteration process, respectively.

In 2014, Abbas and Nazir [1] introduced a new iteration process and proved that it is faster than all of Picard, Mann and Agarwal et al. processes as follows: For M a convex subset of normed space X and a nonlinear mapping T of M into itself, the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in M is defined by

$$\begin{cases} x_1 = x \in M, \\ x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$.

In 2017, Pant and Shukla [17] introduced a new type of monotone nonexpansive mappings in an ordered Banach space X with partial order $\le$. This new class of nonlinear mappings properly contains nonexpansive, firmly-nonexpansive and Suzuki-type generalized nonexpansive mappings and partially extends α-nonexpansive mappings as follows: Let X be a Banach space and M be a nonempty subset of X. A mapping $T : M \to M$ is said to be *generalized α-nonexpansive*, if there exists $\alpha \in [0,1)$ such that

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \Longrightarrow \|Tx - Ty\| \le \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1-2\alpha)\|x - y\|$$

for all $x, y \in M$. Clearly, generalized 0-nonexpansive maps is exactly Suzuki-generalized nonexpansive maps. The next simple example can show this fact. We see that, let $M = \{(0,0),(2,0),(0,4),(4,0),(4,5),(5,4)\}$ be a subset of $\mathbb{R}$ with dictionary order. Define a norm $\|\cdot\|$ on M by $\|(x_1,x_2)\| = |x_1| + |x_2|$. Then $(X, \|\cdot\|)$ is a Banach space. Define a mapping $T : M \to M$ by $T(0,0) = (0,0)$, $T(2,0) = (0,0)$, $T(0,4) = (0,0)$, $T(4,0) = (2,0)$, $T(4,5) = (4,0)$, $T(5,4) = (0,4)$. Then, T is a generalized α-nonexpansive mapping for $\alpha \ge \frac{1}{5}$, but is neither a Suzuki-generalize nonexpansive nor an a-nonexpansive mapping (see [17]).

In 2018, Mebawondu and Izuchukwu [14] introduced and studied some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces. They further established strong and Δ-convergence theorems for Picard Normal S-iteration scheme generated by a generalized α-nonexpansive mapping in the frame work of uniformly convex hyperbolic spaces. A hyperbolic space is a triple (X, d, W), where (X, d) is a metric space and $W : X^2 \times [0,1] \to X$ is such that

(W1) $d(u, W(x,y,\alpha)) \le \alpha d(u,x) + (1-\alpha)d(u,y)$;

(W2) $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta| d(x,y)$;

(W3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(W4) $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y)+\alpha d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$. The setting of hyperbolic spaces introduced by Kohlenbach [10]. The class of hyperbolic spaces contains normed spaces and convex subsets thereof, the Hilbert ball equipped with the hyperbolic metric. All normed spaces and their subsets are the examples of hyperbolic spaces as well convex metric spaces. It is remarked that CAT(0) spaces and Banach spaces are important examples of this type of hyperbolic spaces.

In this paper, we introduce and study some properties of the generalized α-nonexpansive mapping on a nonempty subset of a hyperbolic space and prove fixed point theorems for generalized α-nonexpansive mappings, Δ-convergence theorems and convergence theorems in a hyperbolic space.

2. Preliminaries

Now, we recall definitions on hyperbolic spaces. If $x,y \to X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for the more general setting of convex metric space [21], as follows:

$$d(x,W(x,y,\lambda)) = \lambda d(x,y) \quad \text{and} \quad d(y,W(x,y,\lambda)) = (1-\lambda)d(x,y)$$

for all $x,y \in X$ and $\lambda \in [0,1]$.

A hyperbolic space (X,d,W) is uniformly convex [18] if for any $r>0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$,

$$d\left(W\left(x,y,\frac{1}{2}\right),a\right) \leq (1-\delta)r$$

provided $d(x,a) \leq r, d(y,a) \leq r$ and $d(x,y) \geq \varepsilon r$.

A mapping $\eta : (0,1) \times (0,2] \to (0,1]$, which providing such a $\delta = \eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon \in (0,2]$, is called as a modulus of uniform convexity [19]. We call the function η is monotone if it decreases with r (for fixed ε), that is, $\eta(r_2,\varepsilon) \leq \eta(r_1,\varepsilon)$, for all $r_2 \geq r_1 > 0$.

Let M be a nonempty subset of metric space (X,d) and $\{x_n\}$ be any bounded sequence in X while $\operatorname{diam}(M)$ denote the diameter of M.

Definition 2.1. Let M be a nonempty subset of metric space X and let $\{x_n\}$ be any bounded sequence in M. Let a continuous functional $r_a(\cdot,\{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad \text{for all } x \in X.$$

Then, consider the following:

(i) The infimum of $r_a(\cdot,\{x_n\})$ over M is said to be the *asymptotic radius* of $\{x_n\}$ with respect to M and is denoted by $r_a(M,\{x_n\})$;

(ii) a point $z \in M$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to M if

$$r_a(z,\{x_n\}) = \inf r_a(x,\{x_n\}), x \in M$$

the set of all asymptotic centers of $\{x_n\}$ with respect to M is denoted by $A(M)(M,\{x_n\})$;

(iii) this set may be empty, a singleton, or certain infinitely many points;

(iv) if the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $A(M)(X,\{x_n\}) = A(M)(\{x_n\})$, respectively;

(v) for $x \in X, r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces (see [5]).

Definition 2.2 ([9])**.** A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

Remark 2.3 ([11])**.** We note that Δ-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.4 ([12])**.** *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset M of X.*

Lemma 2.5 ([4])**.** *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_n\}) = \{x_1\}$ and $\{d(x_n,x_1)\}$ converges, then $x = x_1$.*

Lemma 2.6 ([8])**.** *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x) \le c$, $\limsup_{n\to\infty} d(y_n,x) \le c$ and $\lim_{n\to\infty} Wd(x_n,y_n,\alpha_n)$ for some $c \ge 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

Definition 2.7. Let M be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to M if for all $x \in M$ and $n \ge 1$,

$$d(x_{n+1},x) \le d(x_n,x).$$

Example 2.8. Let M be a nonempty subset of X, and $T : M \to M$ be a quasi-nonexpansive (in particular, nonexpansive) mapping such that $F(T) \neq \emptyset$ and $x_0 \in M$. Then the sequence $\{x_n\}$ of Picard iterates is Fejér monotone with respect to $F(T)$.

Proposition 2.9 ([5])**.** *Let $\{x_n\}$ be a sequence in X and M be a nonempty subset of X. Suppose that $\{x_n\}$ is Fejér monotone with respect to M, then we have the followings:*

(1) *$\{x_n\}$ is bounded;*

(2) *The sequence $\{d(x_n,p)\}$ is decreasing and converges for all $p \in F(T)$;*

(3) *$\lim_{n\to\infty} d(x_n,F(T))$ exists.*

Definition 2.10 ([16])**.** Let M be a nonempty subset of a metric space X. A self mapping T of M with nonempty fixed point set $F(T)$ in M is said to satisfy Condition I if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$, such that $d(x,Tx) \ge f(D(x,F(T)))$ for all $x \in M$, where $D(x,F(T)) = \inf\{d(x,p) : p \in F(T)\}$.

3. Main Results

In this section, we will prove some property for class of generalized α-nonexpansive mappings in a hyperbolic spaces. First, we introduce generalized α-nonexpansive mappings in a hyperbolic space as follows: Let M be a nonempty subset of hyperbolic space X. Then $T : M \to M$ is said to satisfy *generalized α-nonexpansive*, if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \alpha d(y,Tx) + \alpha d(x,Ty) + (1-2\alpha)d(x,y)$$

for all $x, y \in M$.

From [17, Proposition 3.5, Lemma 3.7 and Lemma 3.8], we introduce Lemma 3.1, Lemma 3.2 and Lemma 3.3 in hyperbolic space respectively.

Lemma 3.1. *Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in M$:*

(i) $d(Tx,T^2x) \le d(x,Tx)$;

(ii) *Either* $\frac{1}{2}d(x,Tx) \le d(x,y)$ *or* $\frac{1}{2}d(Tx,T^2x) \le d(Tx,y)$;

(iii) *Either* $d(Tx,Ty) \le \alpha(Tx,y) + \alpha d(x,Ty) + (1-2\alpha)d(x,y)$
or $d(T^2x,Ty) \le \alpha d(Tx,Ty) + \alpha d(T^2x,y) + (1-2\alpha)d(Tx,y)$.

Proof. (i) Since,

$$\frac{1}{2}d(x,Tx) \le d(x,Tx)$$

by definition of T, we obtain that

$$\begin{aligned} d(Tx,T^2x) &\le \alpha d(Tx,Tx) + \alpha d(T^2x,x) + (1-2\alpha)d(x,Tx) \\ &= \alpha d(T^2x,x) + (1-2\alpha)d(x,Tx). \end{aligned}$$

We choose $\alpha = 0 < 1$, then we have $d(Tx,T^2x) \le d(x,Tx)$.
(ii) We will prove by contradiction, suppose that

$$\frac{1}{2}d(x,Tx) > d(x,y) \quad \text{and} \quad \frac{1}{2}d(Tx,T^2x) > d(Tx,y).$$

So, by (i) we have

$$\begin{aligned} d(x,Tx) &\le d(x,y) + d(Tx,y) \\ &< \frac{1}{2}d(x,Tx) + \frac{1}{2}d(Tx,T^2x) \\ &\le d(x,Tx). \end{aligned}$$

This is a contradiction. Hence, we have $\frac{1}{2}d(x,Tx) \le d(x,y)$ or $\frac{1}{2}d(Tx,T^2x) \le d(Tx,y)$. (iii) follows from (ii). □

Lemma 3.2. *Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in M$ with $x \le y$,*

$$d(x,Tx) \le \frac{(3+\alpha)}{(1-\alpha)}d(x,Tx) + d(x,y).$$

Proof. By Lemma 3.1, we have for all $x, y \in M$ either

$$d(Tx,Ty) \le \alpha(Tx,y) + \alpha d(x,Ty) + (1-2\alpha)d(x,y)$$

or

$$d(T^2x,Ty) \le \alpha d(Tx,Ty) + \alpha d(T^2x,y) + (1-2\alpha)d(Tx,y).$$

In first case, we consider

$$\begin{aligned} d(x,Ty) &\le d(x,Tx) + d(Tx,Ty) \\ &\le d(x,Tx) + \alpha d(Tx,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y) \\ &\le d(x,Tx) + \alpha d(Tx,x) + \alpha d(x,y) + \alpha d(Ty,x) + (1-2\alpha)d(x,y). \end{aligned}$$

This implies that

$$d(x,Ty) \le \frac{(1+\alpha)}{(1-\alpha)} d(Tx,x) + d(x,y)$$

In other case, we consider

$$\begin{aligned} d(x,Ty) &\le d(x,Tx) + d(Tx,T^2x) + d(T^2x,Ty) \\ &\le 2d(x,Tx) + \alpha d(Tx,Ty) + \alpha d(T^2,y) + (1-2\alpha)d(Tx,y) \\ &\le 2d(x,Tx) + \alpha d(Tx,x) + \alpha d(Ty,x) + \alpha d(T^2x,Tx) + \alpha d(Tx,y) + (1-2\alpha)d(Tx,y) \\ &\le (2+\alpha)d(x,Tx) + \alpha d(Ty,x) + \alpha d(x,Tx) + (1-\alpha)d(Tx,y) \\ &\le (2+\alpha)d(x,Tx) + \alpha d(Ty,x) + \alpha d(x,Tx) + (1-\alpha)d(Tx,x) + (1-\alpha)d(x,y). \end{aligned}$$

This implies that

$$d(x,Ty) \le \frac{(3+\alpha)}{(1-\alpha)} d(x,Tx) + d(x,y).$$

□

Lemma 3.3. *Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping and $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping.*

Proof. Let $p \in F(T)$ and $x \in M$. Since $\frac{1}{2}d(z,Tz) = 0 \le d(z,x)$, we obtain that

$$\begin{aligned} d(p,Tx) &= d(Tp,Tx) \\ &\le \alpha d(Tp,x) + \alpha d(Tx,p) + (1-2\alpha)d(p,x). \end{aligned}$$

We choose $\alpha = 0 < 1$, then we have

$$d(p,Tx) \le d(p,x).$$

Hence, T is a quasi-nonexpansive mapping. □

Lemma 3.4. *Let X be complete uniformly convex hyperbolic space with monotone modulus of convexity η, M be a nonempty closed convex subset of X and T be a self generalized α-nonexpansive mapping on M. If $\{x_n\}$ is bounded sequence in M such that*

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0,$$

then T has a fixed point.

Proof. Since $\{x_n\}$ is bounded sequence in X, and by Lemma 2.4, we have $\{x_n\}$ has unique asymptotic center in M and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T satisfies the generalized α-nonexpansive and from Lemma 3.2 such that

$$d(x_n, Tx) \le \frac{(3+\alpha)}{(1-\alpha)} d(x_n, Tx_n) + d(x_n, x).$$

Now, we take $\limsup$ as $n \to \infty$ both the sides, we have

$$\begin{aligned} r_a(Tx, \{x_n\}) &= \limsup_{n\to\infty} d(x_n, Tx) \\ &\le \limsup_{n\to\infty} \left[\frac{(3+\alpha)}{(1-\alpha)} d(x_n, Tx_n) + d(x_n, x)\right] \\ &\le \limsup_{n\to\infty} d(x_n, x) = r_a(x, \{x_n\}). \end{aligned}$$

By the uniqueness of asymptotic center, $Tx = x$, this implies that x is fixed point of T. Hence, T has a fixed point. □

Lemma 3.5. *Let M be a nonempty and convex subset of a strictly convex hyperbolic space X. Let T be a self generalized α-nonexpansive mapping on M and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Proof. Assume that $\{x_n\} \subseteq F(T)$ such that $\{x_n\}$ converges to some $y \in M$. We will show that $y \in F(T)$. By Lemma 3.2, we get that

$$d(x_n, Ty) \le \frac{(3+\alpha)}{(1-\alpha)} d(x_n, Tx_n) + d(x_n, y),$$

taking $\limsup$ as $n \to \infty$ both the sides, we have

$$\limsup_{n\to\infty} d(x_n, Ty) \le \limsup_{n\to\infty} \frac{(3+\alpha)}{(1-\alpha)} d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, y).$$

So, $\limsup_{n\to\infty} d(x_n, Ty) \le \limsup_{n\to\infty} d(x_n, y)$. By the uniqueness of the limit point of M, we obtain that $Ty = y$. Therefore, $F(T)$ is closed.

Next, we show that $F(T)$ is convex, let $x, y \in F(T)$, then for $\beta \in [0,1]$, we have

$$\begin{aligned} d(x, T(W(x,y,\beta))) &\le \frac{(3+\alpha)}{(1-\alpha)} d(x, Tx) + d(x, W(x,y,\beta)) \\ &\le d(x, W(x,y,\beta)) \end{aligned}$$

and

$$\begin{aligned} d(y, T(W(x,y,\beta))) &\le \frac{(3+\alpha)}{(1-\alpha)} d(y, Ty) + d(y, W(x,y,\beta)) \\ &\le d(y, W(x,y,\beta)) \end{aligned}$$

Now, we consider

$$\begin{aligned} d(x,y) &\le d(x, T(W(x,y,\beta))) + d(T(W(x,y,\beta)), y) \\ &\le d(x, W(x,y,\beta)) + d(W(x,y,\beta), y) \\ &\le d(x,y). \end{aligned}$$

Therefore, if $d(x, T(W(x,y,\beta))) < d(x, W(x,y,\beta))$ or $d(T(W(x,y,\beta)), y) < d(W(x,y,\beta), y)$, then which the contradiction to $d(x,y) < d(x,y)$, so $d(x, T(W(x,y,\beta))) = d(x, W(x,y,\beta))$ and

$d(T(W(x,y,\beta)),y) = d(W(x,y,\beta),y)$. Since M is strictly convex, we have $T(W(x,y,\beta) = W(x,y,\beta)$, that is $W(x,y,\beta) \in F(T)$. Hence, $F(T)$ is convex. □

Theorem 3.6. *Let M be a nonempty closed bounded and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and T be a self generalized α-nonexpansive mapping on M. Suppose that $\{x_n\}$ is a sequence in M, with $d(x_n,Tx_n) \to 0$. If $A(M)(M,\{x_n\}) = x$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Proof. Suppose that there exists some approximate fixed point sequence $\{x_n\}$. By Lemma 2.4, the asymptotic center of any bounded sequence is in M has a unique asymptotic center in M. Let $A(M)(M,\{x_n\}) = x$. We will prove that $x = Tx$. From Lemma 3.2, we have

$$d(x_n,Tx) \le \frac{(3+\alpha)}{(1-\alpha)} d(x_n,Tx_n) + d(x_n,x),$$

taking lim sup as $n \to \infty$ both the sides, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n,Tx) &\le \frac{(3+\alpha)}{(1-\alpha)} \limsup_{n\to\infty} d(x_n,Tx_n) + \limsup_{n\to\infty} d(x_n,x)\\ &= \limsup_{n\to\infty} d(x_n,x).\end{aligned}$$

By uniqueness of the asymptotic center implies $Tx = x$. Moreover, $F(T)$ closed and convex, by the prove in Lemma 3.5. □

Corollary 3.7. *Let M be a nonempty closed bounded and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in M, with $d(x_n,Tx_n) \to 0$. If T satisfies generalized α-nonexpansive and $A(M)(M,\{x_n\}) = x$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Now, we expand the result of Abbas and Nazir [1] to generalized α-nonexpansive mappings in hyperbolic spaces, as follows: Let M be a nonempty closed convex subset of a hyperbolic space X and T be a self generalized α-nonexpansive mapping on M. For any $x_1 \in M$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = W(Ty_n, Tz_n, \alpha_n) \\ y_n = W(z_n, Tz_n, \beta_n) \\ z_n = W(x_n, Tx_n, \gamma_n) \; n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0,1]$ for all $n \in \mathbb{N}$.

Lemma 3.8. *Let M be a nonempty closed convex subset of a hyperbolic space X and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive. If $\{x_n\}$ is a sequence defined by* (3.1)*, then $\{x_n\}$ is Fejér monotone with respect to $F(T)$.*

Proof. Since T satisfies the generalized α-nonexpansive and $p \in F(T)$, we have

$$\frac{1}{2} d(p,Tp) = 0 \le d(p,x_n),$$

$$\frac{1}{2}d(p,Tp) = 0 \le d(p,y_n)$$

and

$$\frac{1}{2}d(p,Tp) = 0 \le d(p,z_n),$$

for all $n \in \mathbb{N}$. We obtain that

$$d(Tp,Tx_n) \le \alpha d(Tp,x_n) + \alpha d(Tx_n,p) + (1-2\alpha)d(p,x_n),$$

$$d(Tp,Ty_n) \le \alpha d(Tp,y_n) + \alpha d(Ty_n,p) + (1-2\alpha)d(p,y_n)$$

and

$$d(Tp,Tz_n) \le \alpha d(Tp,z_n) + \alpha d(Tz_n,p) + (1-2\alpha)d(p,z_n).$$

By (3.1) and Lemma 3.3, we have

$$d(Tp,Tx_n) \le d(p,x_n),$$

$$d(Tp,Ty_n) \le d(p,y_n)$$

and

$$d(Tp,Tz_n) \le d(p,z_n). \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned} d(x_{n+1},p) &= d(W(Ty_n,Tz_n,\alpha_n),p) \\ &\le (1-\alpha_n)d(Ty_n,p) + \alpha_n d(Tz_n,p) \\ &\le (1-\alpha_n)d(y_n,p) + \alpha_n d(z_n,p), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} d(y_n,p) &= d(W(z_n,Tz_n,\beta_n),p) \\ &\le (1-\beta_n)d(z_n,p) + \beta_n d(Tz_n,p) \\ &\le (1-\beta_n)d(z_n,p) + \beta_n d(z_n,p) \\ &= d(z_n,p), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d(z_n,p) &= d(W(x_n,Tx_n,\gamma_n),p) \\ &\le (1-\beta_n)d(x_n,p) + \gamma_n d(Tx_n,p) \\ &\le (1-\beta_n)d(x_n,p) + \gamma_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.5}$$

Now, taking (3.4) in (3.3), we have

$$\begin{aligned} d(x_{n+1},p) &\le (1-\alpha_n)d(z_n,p) + \alpha_n d(z_n,p) \\ &= d(z_n,p). \end{aligned} \tag{3.6}$$

Taking (3.5) in (3.6), we obtain that

$$d(x_{n+1},p) \le d(z_n,p) \le d(x_n,p) \tag{3.7}$$

Hence, $\{x_n\}$ is Fejér monotone with respect to $F(T)$. □

Lemma 3.9. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and T be a self generalized α-nonexpansive mapping on M. If $\{x_n\}$ is a sequence defined by* (3.1)*, then $F(T)$ is nonempty if and only if the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,Tx_n)=0$.*

Proof. Assume that $F(T)$ is nonempty and let $p\in F(T)$. From Lemma 3.8 and Proposition 2.9, we have $\{x_n\}$ is Fejér monotone with respect to $F(T)$ and bounded such that $\lim_{n\to\infty} d(x_n,p)$ exists, let $\lim_{n\to\infty} d(x_n,p)=k$. We divide into two case

(i) If $k=0$, we have

$$d(x_n,Tx_n)\le d(x_n,p)+d(p,Tx_n),$$

by Lemma 3.3, we get

$$d(x_n,Tx_n)\le 2d(x_n,p).$$

Taking lim as $n\to\infty$ on both the sides above inequality, we have

$$\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

(ii) If $k>0$, let $p\in F(T)$ and since T satisfies the generalized α-nonexpansive, by Lemma 3.3, we have

$$d(Tx_n,p)\le d(x_n,p),$$

by taking lim sup as $n\to\infty$ both the sides, we have

$$\limsup_{n\to\infty} d(Tx_n,p)\le k. \tag{3.8}$$

Taking lim sup as $n\to\infty$ both the sides in (3.5), we obtain that

$$\limsup_{n\to\infty} d(z_n,p)\le k. \tag{3.9}$$

From (3.6), we get

$$d(x_{n+1},p)\le d(z_n,p),$$

so, taking lim inf as $n\to\infty$ both the sides, we obtain that

$$\begin{aligned}\liminf_{n\to\infty} d(x_{n+1},p)&\le \liminf_{n\to\infty} d(z_n,p)\\ k&\le \liminf_{n\to\infty} d(z_n,p)\end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we have

$$\lim_{n\to\infty} d(z_n,p)=k,$$

which implies that

$$\begin{aligned}k&=\limsup_{n\to\infty} d(z_n,p)\\ &=\limsup_{n\to\infty} d(W(x_n,Tx_n,\gamma_n),p)\\ &\le \limsup_{n\to\infty}[(1-\gamma_n)d(x_n,p)+\gamma_n d(Tx_n,p)]\\ &\le \limsup_{n\to\infty}(1-\gamma_n)d(x_n,p)+\limsup_{n\to\infty}\gamma_n d(Tx_n,p)=k.\end{aligned}$$

Therefore, by Lemma 2.6, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Hence, from Lemma 3.4, we have $Tx = x$, that is $F(T)$ is nonempty. □

Theorem 3.10. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ satisfies the generalized α-nonexpansive, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in* (3.1)*, Δ-converges to a common fixed point of T.*

Proof. By Lemma 3.8, we have $\{x_n\}$ is a bounded sequence then, $\{x_n\}$ has a Δ-convergent subsequence.

Next, we show that every Δ-convergent subsequence of $\{x_n\}$ has unique Δ-limit $F(T)$. Let u and v Δ-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$. By Lemma 2.4, $A(M)(M,\{u_n\}) = \{u\}$ and $A(M)(M,\{v_n\}) = \{v\}$. By Lemma 3.9, we get

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0.$$

From Lemma 3.4, we have u and v are fixed points of T.

Now, we will show that $u = v$. Assume that $u \neq v$, then by uniqueness of asymptotic center we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, u) &= \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v) \\
&< \limsup_{n\to\infty} d(v_n, u) \\
&= \limsup_{n\to\infty} d(x_n, u),
\end{aligned}$$

which is a contradiction, therefore $u = v$. Hence, the sequence $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof. □

Theorem 3.11. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by* (3.1)*, converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} D(x_n, F(T)) = 0$, where $D(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Proof. Assume that $\{x_n\}$ converges to $p \in F(T)$. Thus, $\lim_{n\to\infty} d(x_n, p) = 0$, since $0 \le D(x_n, F(T) \le d(x_n, p) \le 0$. Hence, $\liminf_{n\to\infty} D(x_n, F(T)) = 0$.

Conversely, from Lemma 3.5, we have $F(T)$ is closed. Assume that

$$\liminf_{n\to\infty} D(x_n, F(T)) = 0.$$

From (3.7), we have

$$D(x_{n+1},F(T)) \le D(x_n,F(T)), \quad n \in \mathbb{N}$$

then by Lemma 3.8 and Proposition 2.9, we obtain that $\lim_{n\to\infty} d(x_n,F(T))$ exists. Then we have $\lim_{n\to\infty} D(x_n,F(T)) = 0$.

Now, we will show that $\{x_n\}$ is convergent to $p \in F(T)$. Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ we get

$$d(x_{n_k},p_k) < \frac{1}{2^k},$$

for all $k \ge 1$ where $\{p_k\}$ is in $F(T)$. By Lemma 3.8, we have

$$d(x_{n_{k+1}},p_k) \le d(x_{n_k},p_k) < \frac{1}{2^k},$$

this implies that

$$\begin{aligned} d(p_{k+1},p_k) &\le d(p_{k+1},x_{n_{k+1}}) + d(x_{n_{k+1}},p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed, $\{p_k\}$ is a convergent sequence. Let $\{p_k\}$ converges to p. Since

$$d(x_{n_k},p) \le d(x_{n_k},p_k) + d(p_k,p) \to 0, \text{ as } k \to \infty,$$

such that $\lim_{k\to\infty} d(x_{n_k},p) = 0$. Since $\lim_{n\to\infty} d(x_n,p)$ exists, the sequence $\{x_n\}$ is convergent to p. This completes the proof. □

Theorem 3.12. *Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive. Moreover, T satisfies the condition I with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ which is defined by* (3.1)*, converges strongly to some fixed point of T.*

Proof. From Lemma 3.5, we have $F(T)$ is closed. Observe that by Lemma 3.8, we have $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. It follows from the condition I that

$$\lim_{n\to\infty} f(D(x_n,F(T))) \le \lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

Thus, we get

$$\lim_{n\to\infty} f(D(x_n,F(T))) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing mapping satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n,F(T)) = 0$. Rest of the proof follows in lines of Theorem 3.11. Hence the sequence $\{x_n\}$ is convergent to $p \in F(T)$. This completes the proof. □

4. Conclusion

In this paper, we studied some properties of the generalized α-nonexpansive mappings on a nonempty subset of a hyperbolic space, proved fixed point theorems for generalized α-nonexpansive mappings and proved convergence theorems. Moreover, we obtain that corollary for the generalized α-nonexpansive mappings on a nonempty subset of a hyperbolic space as follows:

(1) Let M be a nonempty subset of hyperbolic space X and $T : M \to M$ be a generalized α-nonexpansive mapping. Then, for all $x, y \in M$:

 (i) $d(Tx, T^2x) \le d(x, Tx)$;

 (ii) Either $\frac{1}{2}d(x, Tx) \le d(x, y)$ or $\frac{1}{2}d(Tx, T^2x) \le d(Tx, y)$;

 (iii) Either $d(Tx, Ty) \le \alpha(Tx, y) + \alpha d(x, Ty) + (1-2\alpha)d(x, y)$ or $d(T^2x, Ty) \le \alpha d(Tx, Ty) + \alpha d(T^2x, y) + (1-2\alpha)d(Tx, y)$.

(2) Let M be a nonempty closed bounded and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and T be a self generalized α-nonexpansive mapping on M. Suppose that $\{x_n\}$ is a sequence in M, with $d(x_n, Tx_n) \to 0$. If $A(M)(M, \{x_n\}) = x$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

(3) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : M \to M$ satisfies the generalized α-nonexpansive, such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (3.1), Δ-converges to a common fixed point of T.

(4) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} D(x_n, F(T)) = 0$, where $D(x_n, F(T)) = \inf_{x \in F(T)} d(x_n, x)$.

(5) Let M be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $T : M \to M$ be a mapping which satisfies the generalized α-nonexpansive. Moreover, T satisfies the condition I with $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of T.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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