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Research Article

Differential Transform Method in General Orthogonal Curvilinear Coordinates

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Abstract. Many real world problems are governed by non-linear differential equations. These may be single or system of ordinary or partial differential equations. In practice, problems involving single differential equations are mostly solved using analytical procedures, while those with systems of equations are solved numerically. A semi-analytical procedure namely *Differential Transform Method* (DTM) obtained from Taylor series in Cartesian co-ordinates is being used to solve linear or non-linear equations in practice. This paper introduces the Taylor series and DTM for general orthogonal curvilinear co-ordinates and focuses mainly on DTM in standard two-dimensional polar co-ordinates and three-dimensional cylindrical polar and spherical polar coordinates.

Keywords. Taylor series; Differential transform method; General orthogonal curvilinear coordinates

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1. Introduction

Taylor series is a series representation of a function with its terms obtained from the values of its derivatives at a point. Taylor series was introduced by the English Mathematician Brook Taylor in 1715. Maclaurin series, named after the Scottish Mathematician Colin Maclaurin is a special case of Taylor series. The Taylor series method expresses the solution of a differential equation as a power series expansion. It is used as a tool to numerically solve the initial value problems. Taylor series forms the base for several numerical methods under various finite difference schemes.

Differential Transform Method (DTM) is a modified form of the Taylor series method. It is currently used as a technique for analytically calculating the power series of the solution in terms of the initial value parameters. A continuous differentiable function $f(x)$ can be expanded in a Taylor series about the point $x = x_0$. A semi-analytical approach namely DTM has been used by many researchers for solving different problems involving ordinary differential equations or partial differential equations. DTM was first proposed by Pukhov [6]. Zhou [10] used DTM to solve both linear and non-linear initial value problems in electric circuit analysis. Different aspects of DTM have been analyzed and developed by many researchers. During the last decade, significant progress has been made in application of the DTM to some linear and non-linear initial value problems. Chen and Ho [2] have introduced the two-dimensional DTM and used it to solve partial differential equations. Bert [1] has analyzed the application of DTM to heat conduction in tapered fins. Kurnaz *et al*. [4] have developed n-dimensional DTM and investigated its application to differential equations. Erturk and Momani [3] have solved two-point non-linear boundary value problems for obtaining positive solutions by DTM. Ravi Kanth and Aruna [7] have used DTM to solve singular two-point boundary value problems. Application of DTM in fluid flow problems and DTM in fractional order have been analyzed by Rashidi and Sadri [8], Rashidi *et al*. [9], and Odibat *et al*. [5].

In this paper, the authors present the DTM in general orthogonal curvilinear coordinates. This requires the knowledge of Taylor series expansion of continuously differentiable functions.

Assuming $f(\vec{r})$ to be such a function and taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$, $\delta\vec{r} = \delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k}$, $\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ and noting that $\delta\vec{r}\vec{\nabla} \equiv \delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y} + \delta z\frac{\partial}{\partial z}$, Taylor series in the vector form may be given as

$$f(\vec{r}) = f(\vec{r}_0) + \frac{1}{1!}\left(\delta\vec{r}\vec{\nabla}\right) f|_{(\vec{r}_0)} + \frac{1}{2!}\left(\delta\vec{r}\vec{\nabla}\right)^2 f|_{(\vec{r}_0)} + \ldots + \frac{1}{n!}\left(\delta\vec{r}\vec{\nabla}\right)^n f|_{(\vec{r}_0)} + \ldots \tag{1.1}$$

Having seen the vector form of Taylor series in Section 1, now Section 2 presents the Taylor series in general orthogonal curvilinear coordinates. Then Section 3 introduces the DTM in these coordinates. Section 4 illustrates the above method with simple application problems in standard curvilinear coordinates. Section 5 gives the conclusion of the work.

2. Taylor Series in General Orthogonal Curvilinear Coordinates

Let f be a continuous differentiable function in three dimensional space. Then, using $\vec{\delta r} = h_1\delta\lambda\widehat{a}_1 + h_2\delta\mu\widehat{a}_2 + h_3\delta\nu\widehat{a}_3$, $\vec{\nabla} \equiv \frac{1}{h_1}\frac{\partial}{\partial\lambda}\widehat{a}_1 + \frac{1}{h_2}\frac{\partial}{\partial\mu}\widehat{a}_2 + \frac{1}{h_3}\frac{\partial}{\partial\nu}\widehat{a}_3$ and taking $\vec{r} = \lambda\widehat{a}_1 + \mu\widehat{a}_2 + \nu\widehat{a}_3$ and $\vec{r}_0 = \lambda_0\widehat{a}_1 + \mu_0\widehat{a}_2 + \nu_0\widehat{a}_3$ in the vector form of the Taylor series (1.1), the component form of equation (1.1) in general orthogonal curvilinear co-ordinates is given by

$$f(\lambda,\mu,\nu)=f(\lambda_0,\mu_0,\nu_0)+\frac{1}{1!}\left(h_1\delta\lambda\frac{\partial}{h_1\partial\lambda}+h_2\delta\mu\frac{\partial}{h_2\partial\mu}+h_3\delta\nu\frac{\partial}{h_3\partial\nu}\right)f\Big|_{(\lambda_0,\mu_0,\nu_0)}$$
$$+\frac{1}{2!}\left(h_1\delta\lambda\frac{\partial}{h_1\partial\lambda}+h_2\delta\mu\frac{\partial}{h_2\partial\mu}+h_3\delta\nu\frac{\partial}{h_3\partial\nu}\right)^2 f\Big|_{(\lambda_0,\mu_0,\nu_0)}+\dots . \tag{2.1}$$

For a two-dimensional coordinate system, ν co-ordinate does not exist. If $\lambda=x$, $\mu=y$, $h_1=1$, $h_2=1$, $\widehat{a}_1=\widehat{i}$, $\widehat{a}_2=\widehat{j}$ and $\lambda=x$, $\mu=y$, $\nu=z$, $h_1=1$, $h_2=1$, $h_3=1$, $\widehat{a}_1=\widehat{i}$, $\widehat{a}_2=\widehat{j}$, $\widehat{a}_3=\widehat{k}$, then equation (2.1) reduces to Taylor series in two and three dimensional Cartesian coordinates, respectively.

3. DTM in General Orthogonal Curvilinear Coordinates

Let the Taylor series in general orthogonal curvilinear coordinates in equation (2.1) be rewritten as

$$f(\lambda,\mu,\nu)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}(\lambda-\lambda_0)^k(\mu-\mu_0)^h(\nu-\nu_0)^l\,\frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}}{\partial\lambda^k\partial\mu^h\partial\nu^l}f(\lambda,\mu,\nu)\right]_{\lambda=\lambda_0,\mu=\mu_0,\nu=\nu_0}. \tag{3.1}$$

Then, the differential transform of f about (λ_0,μ_0,ν_0) denoted by $Df(\lambda,\mu,\nu)$ is defined as

$$Df(\lambda,\mu,\nu)=F(k,h,l)=\frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}}{\partial\lambda^k\partial\mu^h\partial\nu^l}f(\lambda,\mu,\nu)\right]_{\lambda=\lambda_0,\mu=\mu_0,\nu=\nu_0} \tag{3.2}$$

and its inverse transform is defined as

$$D^{-1}F(k,h,l)=f(\lambda,\mu,\nu)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}(\lambda-\lambda_0)^k(\mu-\mu_0)^h(\nu-\nu_0)^lF(k,h,l). \tag{3.3}$$

Some properties of DTM in general orthogonal curvilinear coordinates are given below:

(I) If $f(\lambda,\mu,\nu)$and $g(\lambda,\mu,\nu)$ are two continuous differentiable functions, then

$$f(\lambda,\mu,\nu)\pm g(\lambda,\mu,\nu)\xrightarrow{\text{DTM}}F(k,h,l)\pm G(k,h,l).$$

The algebraic sum and difference of the functions f and g are transformed into the algebraic sum and difference of their images

(II) $cf(\lambda,\mu,\nu)\xrightarrow{\text{DTM}}cF(k,h,l)$.

When the function $f(\lambda,\mu,\nu)$ is multiplied by a constant c, it is transformed into the image multiplied by the same constant.

(III) The product of the functions $f(\lambda,\mu,\nu)$ and $g(\lambda,\mu,\nu)$ is transformed into the algebraic convolution of their images.

$$f(\lambda,\mu,\nu)\times g(\lambda,\mu,\nu)\xrightarrow{\text{DTM}}\sum_{a=0}^{k}\sum_{b=0}^{h}\sum_{c=0}^{l}F(a,h-b,l-c)G(k-a,b,c).$$

(IV) $\dfrac{\partial^m}{\partial\lambda^m}f(\lambda,\mu,\nu)\xrightarrow{\text{DTM}}\dfrac{(k+m)!}{k!}F(k+m,h,l)$. (V) $\lambda^m f(\lambda,\mu,\nu)\xrightarrow{DTM}F(k-m,h,l)$.

By taking $\lambda=x$, $\mu=y$ and $\lambda=x$, $\mu=y$, $\nu=z$ in equation (3.1) for two and three dimensional Cartesian coordinates, DTM in two and three dimensional Cartesian coordinates can be obtained, respectively.

4. Applications

4.1 Unsteady Flow in a Tube

Consider the unsteady flow of a viscous fluid in a tube in which, besides time as one of the independent variables, the velocity field is a function of only one space coordinate. The governing equations are the Navier Stokes equations reduced to the form

$$u_t(r,t) = u_{rr}(r,t) + \frac{1}{r}u_r(r,t) + P \tag{4.1}$$

subject to the initial condition

$$u(r,0) = 1 - r^2. \tag{4.2}$$

Equation (4.1) can be written as

$$ru_t(r,t) = ru_{rr}(r,t) + u_r(r,t) + rP. \tag{4.3}$$

Taking Differential transform on equations (4.3) and (4.2), we get

$$\begin{aligned}
&\sum_{k_1=0}^{k}\sum_{h_1=0}^{h}\delta(k_1-1)\delta(h-h_1)(h_1+1)U(k-k_1,h_1+1)\\
&\quad -\sum_{k_1=0}^{k}\sum_{h_1=0}^{h}\delta(k_1-1)\delta(h-h_1)(k-k_1+1)(k-k_1+2)U(k-k_1+2,h_1)\\
&\quad -(k+1)U(k+1,h)-\delta(k-1)\delta(h)P = 0
\end{aligned} \tag{4.4}$$

with

$$U(k,0) = \delta(k) - \delta(k-2). \tag{4.5}$$

Taking $h = 0$ in equation (4.4) and varying k, we get

$$U(k,1) = \begin{cases} P-4, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Also, taking $h = 1$ and varying k in equation (4.4) and using (4.6), we get

$$U(k,2) = 0, \quad \text{for all } k. \tag{4.7}$$

Proceeding iteratively, we get

$$U(k,h) = \begin{cases} 1, & \text{if } k = 0 \text{ and } h = 0, \\ -1, & \text{if } k = 2 \text{ and } h = 0, \\ P-4, & \text{if } k = 0 \text{ and } h = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

By taking inverse DTM on equation (4.8), the solution of eq. (4.1) is obtained as

$$\begin{aligned}
u(r,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)r^k t^h \\
&= \sum_{h=0}^{\infty}U(k,0)r^k + \sum_{h=0}^{\infty}U(k,1)r^k t + \sum_{h=0}^{\infty}U(k,2)r^k t^2 + \dots \\
&= 1 - r^2 + (P-4)t
\end{aligned}$$

which coincides with the exact solution of the given problem.

4.2 Laplace Equation in Polar Coordinates

In this subsection, we consider the Laplace equation in polar form with Dirichlet boundary conditions as given below

$$u_{rr}(r,\theta)+\frac{1}{r}u_r(r,\theta)+\frac{1}{r^2}u_{\theta\theta}(r,\theta)=0 \tag{4.9}$$

with boundary conditions

$$\left.\begin{array}{lll}\text{(i)} & u(r,0)=0, & \\ \text{(ii)} & u(r,\pi)=0, & \\ \text{(iii)} & u(b,\theta)=T, & 0\le\theta\le\pi.\end{array}\right\} \tag{4.10}$$

Equation (4.9) can be written as

$$r^2 u_{rr}(r,\theta)+r\,u_r(r,\theta)+u_{\theta\theta}(r,\theta)=0. \tag{4.11}$$

Taking differential transform on equation (4.11)

$$\begin{aligned}&\sum_{k_1=0}^{k}\sum_{h_1=0}^{h}\delta(k_1-2)\delta(h-h_1)(k-k_1+2)(k-k_1+1)U(k-k_1+2,h_1)\\ &\quad+\sum_{k_1=0}^{k}\sum_{h_1=0}^{h}\delta(k_1-1)\delta(h-h_1)(k-k_1+1)U(k-k_1+1,h_1)+(h+1)(h+2)U(k,h+2)=0.\end{aligned} \tag{4.12}$$

Taking differential transform of condition (i) of equation (4.10), we get,

$$U(k,0)=0. \tag{4.13}$$

Now, as we substitute the values for $h=0,1,2,\dots$ in equation (4.12), we get $U(k,2),U(k,3),\dots$ in terms of $U(k,0),U(k,1),\dots$, respectively. So assume

$$u_\theta(r,0)=f(r)=\sum_{n=1}^{\infty}na_nr^n. \tag{4.14}$$

In this, a_n are new unknowns to be determined later using condition (iii) of (4.10). Also

$$F(1)=a_1,F(2)=2a_2,F(3)=3a_3,F(4)=\ldots=F(k)=ka_k,$$

where $F(k)$ is the differential transform of $f(r)$. Hence

$$U(k,1)=ka_k. \tag{4.15}$$

For $h=0$, equation (4.12) becomes

$$\begin{aligned}&\sum_{k_1=0}^{k}\delta(k_1-2)(k-k_1+2)(k-k_1+1)U(k-k_1+2,0)\\ &\quad+\sum_{k_1=0}^{k}\delta(k_1-1)(k-k_1+1)U(k-k_1+1,0)+(1)(2)U(k,2)=0.\end{aligned} \tag{4.16}$$

For various values of k and using eq. (4.13), eq. (4.16) gives

$$U(k,2)=\frac{-k^2}{2}U(k,0)=0. \tag{4.17}$$

For, $h=1$, eq. (4.16) becomes

$$\sum_{k_1=0}^{k}\delta(k_1-2)(k-k_1+2)(k-k_1+1)U(k-k_1+2,1)$$

$$+\sum_{k_1=0}^{k}\delta(k_1-1)(k-k_1+1)U(k-k_1+1,1)+(6)U(k,3)=0. \tag{4.18}$$

Taking various values of k in equation (4.18) and using equation (4.15), we get

$$U(k,3)=\frac{-k^3}{3!}a_k. \tag{4.19}$$

For $h=2$, equation (4.12) becomes

$$\sum_{k_1=0}^{k}\delta(k_1-2)(k-k_1+2)(k-k_1+1)U(k-k_1+2,2)$$
$$+\sum_{k_1=0}^{k}\delta(k_1-1)(k-k_1+1)U(k-k_1+1,2)+(12)U(k,4)=0. \tag{4.20}$$

For various values of k in equation (4.20) and using equation (4.17), we get

$$U(k,4)=0. \tag{4.21}$$

Continuing the process iteratively, we obtain

$$U(k,h)=\begin{cases}\frac{(-1)^{\frac{h-1}{2}}}{h!}k^h a_k, & h \text{ is odd},\\ 0, & h \text{ is even.}\end{cases} \tag{4.22}$$

Now, by inverse transform, we get

$$u(r,\theta)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}r^k\theta^h U(k,h)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}r^k\theta^h\frac{(-1)^{\frac{h-1}{2}}}{h!}k^h a_k=\sum_{k=0}^{\infty}r^k a_k\sin k\theta. \tag{4.23}$$

Using equation (4.23) and a half range sine series expression in equation (4.10)(iii), we get

$$u(b,\theta)=\sum_{k=0}^{\infty}b^k a_k\sin k\theta=\sum_{k=0}^{\infty}C_k\sin k\theta=T, \tag{4.24}$$

where

$$C_k=\frac{2}{\pi}\int_0^{\pi}T\sin k\theta d\theta=\frac{2T}{\pi}\frac{[1-\cos k\theta]}{k}=\begin{cases}\frac{4T}{k\pi}, & \text{if } k \text{ is odd},\\ 0, & \text{otherwise.}\end{cases} \tag{4.25}$$

Here, $C_k=b^k a_k$. Therefore, $a_k=\frac{C_k}{b^k}$

$$a_k=\begin{cases}\frac{4T}{k\pi b^k}, & \text{if } k \text{ is odd},\\ 0, & \text{otherwise.}\end{cases} \tag{4.26}$$

Hence, (4.23) gives

$$u(r,\theta)=\sum_{k=1,3,5,\ldots}^{\infty}\left(\frac{r}{b}\right)^k\frac{4T}{k\pi}\sin k\theta. \tag{4.27}$$

Equation (4.27) gives the solution from the present method and it matches with the exact solution of the problem.

In the above two problems in polar coordinates, the differential transform method has been applied and the semi analytical solutions have been obtained easily. The method obtains the unknown quantities iteratively which leads to easier computational procedure in such a way that any mathematical software can be applied to solve complex problems involving partial differential equations.

5. Conclusion

Taylor series forms the base for the differential transform method. Now, DTM is widely used in fluid flow problems by many researchers which provides a semi-analytical solution. Fluid flow problems are not only solved in Cartesian coordinates but also in other curvilinear co ordinates for the ease of computation. A semi-analytical procedure of solving fluid flow problems in general orthogonal curvilinear coordinates does not exist in literature. Here, we have presented Taylor series in vector form, which is the base for obtaining Taylor series in general orthogonal curvilinear coordinates. From this, we have derived the DTM in general orthogonal curvilinear coordinates to solve problems in this general coordinate system. The solution method has been illustrated with the help of simple problems for better understanding. The paper portrays that the DTM can be used in any orthogonal curvilinear coordinates, not limited to those described here. Thus any problem related to various disciplines can be easily solved using DTM in the suitable curvilinear coordinate system.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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