Journal of Informatics and Mathematical Sciences
Vol. 10, No. 4, pp. 563–571, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
http://www.rgnpublications.com
DOI: 10.26713/jims.v10i4.1048

Research Article

Homoclinic and Heteroclinic Orbits for a Liénard System

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Abstract. A 2D dynamical system exhibiting a double-zero bifurcation with symmetry of order two is considered. This bifurcation involves the presence in the parameter space of a curve corresponding either to double homoclinic or to heteroclinic bifurcations. In this paper we derive second order approximations for the homoclinic orbits and for the curve of homoclinic bifurcation values considering the system truncated up to five order terms and parameter-dependent coefficients. These approximations were obtained using the regular perturbation method. These formulae are applied to a Liénard system, which develops a double-zero bifurcation with symmetry of order two for some parameters values. Second order approximations for the heteroclinic orbits of this system are also given. The analytical results are very accurate and they are in good accordance with the numerical ones.

Keywords. Double-zero bifurcation; Homoclinic orbit; Heteroclinic orbit; Liénard system

MSC. 37GXX; 37NXX; 34C37

Received: August 1, 2018 **Accepted:** October 14, 2018

This article was submitted to "*5th International Congress on Fundamental and Applied Sciences*" (ICFAS 2018) organized by International University of Balkan, Skopje, Macedonia during June 18-22, 2018.

Academic Editors: Dr. Muhammet Kurulay, Yildiz Technical University, Turkey
Dr. Mehmet Fatih Karaaslan, Yildiz Technical University, Turkey
Dr. Elif Segah Öztaş, Karamanoglu Mehmetbey University, Turkey

1. Introduction

Consider a two-dimensional system

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^2 \tag{1.1}$$

with a smooth f, which has at $\alpha = 0$ the equilibrium $x = 0$ with two zero eigenvalues.

System (1.1) exhibits a double-zero with symmetry of order two bifurcation at $\alpha = 0$ if it is locally topological equivalent around the origin to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 x + \mu_2 y + \sum_{k\geq 1}(a_{2k+1}x^{2k+1} + b_{2k+1}x^{2k}y), \end{aligned} \tag{1.2}$$

where μ_1, μ_2, are real parameters and the coefficients $a_k, b_k \in \mathbb{R}$ satisfy the condition $a_3 b_3 \neq 0$.

Using transformations of time and variables, system (1.2) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \beta_1 x + \beta_2 y + \sigma x^3 - x^2 y + O(|(x,y)|^5), \end{aligned} \tag{1.3}$$

where $\sigma = \pm 1$. System (1.3) is the normal form for this codimension two bifurcation.

In the case $\sigma = -1$ the bifurcation diagram is given in Figure 1 [6]. In this figure, H_1 and H_2 contain Hopf bifurcation values, R^+ and R^- contain pitchfork bifurcation values, B contains saddle-node bifurcation of periodic orbits values and HL contains homoclinic bifurcation values. The first order asymptotic approximation of the curve HL is [6]:

$$HL = \left\{\beta \,\middle|\, \beta_2 = \frac{4}{5}\beta_1 + O(\beta_1^{3/2}),\ \beta_1 > 0\right\}. \tag{1.4}$$

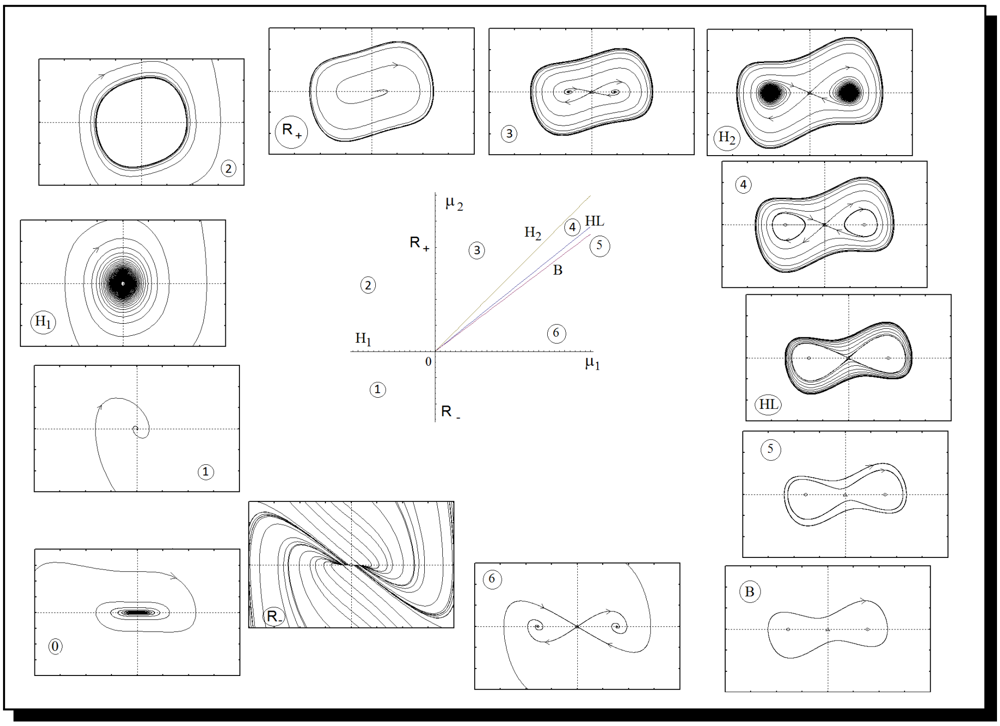

Figure 1. The bifurcation diagram for the double-zero bifurcation, case $\sigma = -1$.

For parameters on the curve HL, system (1.3) has a pair of homoclinic orbits to the origin, which is a saddle point.

In the case $\sigma = 1$ the bifurcation diagram is given in [6], [15]. It involves the presence of a curve of heteroclinic bifurcation values, whose first order asymptotic approximation is:

$$HT = \left\{ \beta \,\middle|\, \beta_2 = -\frac{1}{5}\beta_1 + O(\beta_1^{3/2}),\ \beta_1 < 0 \right\}. \tag{1.5}$$

For parameters on the curve HT, system (1.3) has a pair of heteroclinic orbits connecting two saddle points.

Generally, it is difficult to find analytic expressions for homoclinic and heteroclinic orbits of a dynamical system. Asymptotic methods (such as the elliptic averaging method, the hyperbolic perturbation method, or the regular perturbation method), can be used to detect the presence of these type of orbits, to determine parameter values for which such orbits exist, and to find asymptotic approximations for these orbits. Such results, in various settings, can be found in [2], [3],[4], [5], [8], [9], [10].

Homoclinic orbits bifurcation are involved in specific local bifurcations, as the Bogdanov-Takens bifurcation. Approximations of the homoclinic orbits in the case of the nondegenerated Bogdanov-Takens bifurcation can be found in [1], [12], [13], [17].

For a degenerate Bogdanov-Takens bifurcation, asymptotic approximations of homoclinics are given in our previous paper [14], where the computations were done in a particular case, considering the normal form truncated up to third order terms and with coefficients that do not depend on the parameters. In Section 2 of the present paper we extend the study of homoclinics, considering the normal form (1.2) truncated up to five order terms and parameter-dependent coefficients. In Section 3 we apply the formulae obtained in Section 2 to a Liénard system. In addition to approximations of homoclinic orbits and homoclinic bifurcation values curve for this system, we also give explicit formulae of second order approximations of heteroclinic orbits and heteroclinic bifurcation values curve using results in our paper [15]. Finally, some conclusions are given.

2. Approximations of Homoclinic Orbits

Assume that the coefficients in (1.2) are expressed around the bifurcation value $(\mu_1, \mu_2) = (0,0)$ as functions of the bifurcation parameters μ_1, μ_2 as:

$$\begin{aligned} a_3 &= \sigma a + a_{10}\mu_1 + a_{01}\mu_2 + O(|\mu|^2), \\ b_3 &= -b - b_{10}\mu_1 - b_{01}\mu_2 + O(|\mu|^2), \\ a_5 &= c + O(|\mu|), \\ b_5 &= -d + O(|\mu|), \end{aligned} \tag{2.1}$$

where a, b, c, d, a_{10}, a_{01}, b_{10}, b_{01} are real constants and $a > 0$, $b > 0$, $\sigma = \pm 1$. Thus, system (1.2) reads:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 x + \mu_2 y + \sigma a x^3 - b x^2 y + g\left(\mu_1, \mu_2, x, y\right) + \dots, \end{aligned} \tag{2.2}$$

with $g\left(\mu_1, \mu_2, x, y\right) = (a_{10}\mu_1 + a_{01}\mu_2)x^3 - (b_{10}\mu_1 + b_{01}\mu_2)x^2 y + c x^5 - d x^4 y$.

In our previous paper [14] we treated the particular case when $\sigma=-1$, $a=b=1$, and $g(\mu_1,\mu_2,x,y)\equiv 0$. Obviously the normal form (1.2) was truncated up to order three terms with coefficients a_i, b_i that do not depend on the bifurcation parameters μ_1, μ_2. In [15] we treated the general case when $\sigma=1$, with parameter-dependent coefficients.

In the following we complete our study, for $\sigma=-1$ in the general case, considering the system truncated up to five order terms and parameter-dependent coefficients.

Using the regular perturbation method, we apply the blow-up transformation:

$$\begin{aligned}
&x=\frac{\varepsilon}{\sqrt{a}}\,u, \qquad y=\frac{\varepsilon^2}{\sqrt{a}}\,v,\\
&\mu_1=-\sigma\varepsilon^2, \qquad \mu_2=\frac{b}{a}\varepsilon^2\,\theta
\end{aligned} \tag{2.3}$$

and consider $\varepsilon\ge 0$ and $s=\varepsilon t$ the new time. Thus, system (2.2) becomes:

$$\begin{aligned}
&\frac{du}{ds}=v,\\
&\frac{dv}{ds}=\sigma(-u+u^3)+\frac{b}{a}\varepsilon v(\theta-u^2)+\frac{\varepsilon^2}{a^2}u^3(-\sigma a_{10}a+a_{01}b\theta+cu^2)+\frac{\varepsilon^3}{a^2}u^2v(\sigma b_{10}a-b_{01}b\theta-du^2).
\end{aligned} \tag{2.4}$$

The branch of heteroclinic/homoclinic orbits of system (2.4) parametrized by ε is

$$\left.\begin{aligned}
u&=u_0+\varepsilon u_1+\varepsilon^2u_2+\ldots+\varepsilon^k u_k+\ldots\\
v&=v_0+\varepsilon v_1+\varepsilon^2v_2+\ldots+\varepsilon^k v_k+\ldots\\
\theta&=\theta_0+\varepsilon\theta_1+\varepsilon^2\theta_2+\ldots+\varepsilon^k \theta_k+\ldots
\end{aligned}\right\} \tag{2.5}$$

where k stands for the order of approximation.

For the homoclinic orbits we require that $\lim\limits_{s\to\infty}u(s)=\lim\limits_{s\to-\infty}u(s)=0$ and $\lim\limits_{s\to\infty}v(s)=\lim\limits_{s\to-\infty}v(s)=0$, and that the initial point is situated on the horizontal axis, that is $v(0)=0$.

Replacing (2.5) into (2.4) and collecting the ε^k terms, with $k=0,1,2$, we get a Hamiltonian system and two linear non-homogenous systems, whose solutions are:

$$\left.\begin{aligned}
&u_0(s)=\sqrt{2}\,\mathrm{sech}(s),\\
&v_0(s)=-\sqrt{2}\,\mathrm{sech}(s)\tanh(s),\\
&u_1(s)=\frac{2\sqrt{2}}{5}\frac{b}{a}\ln(\cosh(s))\,\mathrm{sech}(s)\tanh(s),\\
&v_1(s)=-\frac{\sqrt{2}}{5}\frac{b}{a}[1+\cosh(2s)(-1+\ln\cosh(s))-3\ln\cosh(s)]\,\mathrm{sech}^3(s)s,\\
&u_2(s)=\frac{1}{150\sqrt{2}a^2}\,\mathrm{sech}(s)^3(\cosh(2s)(75aa_{10}+60a_{01}b-14b^2+200c-24b^2\ln\cosh(s)\\
&\qquad +12b^2\ln^2\cosh(s)+3(25aa_{10}+20a_{01}b+6b^2+8b^2\ln\cosh(s)\\
&\qquad -12b^2\ln^2\cosh(s)+4b^2s\sinh(2s))),\\
&v_2(s)=u_2'(s).
\end{aligned}\right\} \tag{2.6}$$

Thus we obtain:

Theorem 1. *The second order approximation of a homoclinic orbit in the normal form* (2.2) *is:*

$$x(t) = \frac{\varepsilon}{\sqrt{a}}\left[u_0(\varepsilon t) + \varepsilon u_1(\varepsilon t) + \varepsilon^2 u_2(\varepsilon t)\right],$$
$$y(t) = \frac{\varepsilon^2}{\sqrt{a}}\left[v_0(\varepsilon t) + \varepsilon v_1(\varepsilon t) + \varepsilon^2 v_2(\varepsilon t)\right],$$

where u_0, u_1, u_2 and v_0, v_1, v_2 are given above.

In addition, integrating the system obtained by collecting the ε^3 terms, the terms u_3, v_3 are determined. These terms allow to compute the value θ_2, and thus the second order approximation of HL as follows:

Theorem 2. *The second order approximation of the curve of homoclinic bifurcation values is*

$$HL: \ \mu_2 = \frac{b}{a}\mu_1\left[\theta_0 + \theta_1\sqrt{\mu_1} + \theta_2\mu_1\right],$$

where $\theta_0 = \frac{4}{5}$, $\theta_1 = 0$, and

$$\theta_2 = \frac{1}{13125a^2b}[300a(35a_{10}b+28bb_{01}+40d)+8b\left(1050a_{01}b+23b^2+1600c\right)+10500a^2b_{10}].$$

Remark that in the particular case $a_{10} = a_{01} = 0$, $b_{10} = b_{01} = 0$, $c = d = 0$, $a = b = 1$, we find the result from Theorem 3 in [14].

For the general case $\sigma = 1$, similar results concerning the heteroclinic orbits are given in Theorems 2 and 3 from [15].

3. Homoclinic and Heteroclinic Orbits for a Liénard System

Consider the following generalized Liénard system [11]

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -c_1x_1 + \delta m_0 x_2 - c_3x_1^3 - \delta m_1 x_1^2 x_2 - \delta m_2 x_2^3,\end{aligned} \tag{3.1}$$

where the dot over quantities stands for differentiation with respect to the time τ. This system is invariant with respect to the symmetry $(x_1,x_2) \longrightarrow (-x_1,-x_2)$. As $c_1 = 0$, and $\delta m_0 = 0$, the origin is an equilibrium point with two zero eigenvalues.

In [15], we obtained the normal form of system (3.1) into the form (2.2), with $\mu_1 = -c_1$, $\mu_2 = \delta m_0$, and

$$\begin{aligned}&a = -\sigma c_3, \quad a_{10} = 0, \quad a_{01} = 0,\\ &b = \delta m_1, \quad b_{10} = -3\delta m_2, \quad b_{01} = 0,\\ &c = 0, \quad d = 3c_3\delta m_2.\end{aligned} \tag{3.2}$$

We used a time transformation and the variable transformation

$$\begin{aligned}x &= x_1,\\ y &= (1 - \delta^2 m_0 m_2 x_1^2 + \delta m_2 x_1 x_2)x_2.\end{aligned} \tag{3.3}$$

Thus, a double-zero bifurcation with Z_2-symmetry takes place. Using Theorem 2 above, and Theorem 3 from [15], second order approximations of the curves of homoclinic and heteroclinic bifurcation values involved by this bifurcation are obtained for the Liénard system (3.1), as follows.

Theorem 3. *As $\delta c_3 m_1 \neq 0$, a double-zero bifurcation with symmetry of order two is present at $E_0 = (0,0)$ as $c_1 = 0$, $\delta m_0 = 0$. In addition,*

(i) *if $\delta c_3 m_1 > 0$, the curve of homoclinic bifurcation values is approximated by*

$$HL: m_0 = -\frac{c_1 m_1}{c_3}\left[\frac{4}{5} - \frac{c_1}{13125}\left(\frac{184\delta^2 m_1^2}{c_3^2} + \frac{4500 m_2}{m_1}\right)\right], \tag{3.4}$$

(ii) *if $\delta c_3 m_1 < 0$, the curve of heteroclinic bifurcation values is approximated by*

$$HT: m_0 = -\frac{c_1 m_1}{c_3}\left[\frac{1}{5} + \frac{c_1}{13125}\left(\frac{16\delta^2 m_1^2}{c_3^2} + \frac{4500 m_2}{m_1}\right)\right]. \tag{3.5}$$

Using the local inverse transformation of (3.3), second order approximations of orbits are obtained into the form:

$$\begin{aligned} x_1(t) &= \frac{\varepsilon}{\sqrt{a}}\left[u_0(\varepsilon t) + \varepsilon u_1(\varepsilon t) + \varepsilon^2 u_2(\varepsilon t)\right], \\ x_2(t) &= \frac{\varepsilon^2}{\sqrt{a}}\left[v_0(\varepsilon t) + \varepsilon v_1(\varepsilon t) + \varepsilon^2\left(v_2(\varepsilon t) + \frac{\delta^2 m_0 m_2}{a} u_0^2(\varepsilon t) v_0(\varepsilon t)\right)\right]. \end{aligned} \tag{3.6}$$

Obviously, when the terms containing ε^2 into the right hand side brackets in (3.6) are ignored, the first order approximations are obtained, while as terms containing ε and ε^2 into the right hand side brackets in (3.6) are ignored, then zero order approximations (i.e. Homiltonian) of the orbits are obtained.

Replacing u_i, v_i, $i = 0,1,2$ given by (2.6), into (3.6), and using (3.2), with $\sigma = -1$, the following result is obtained.

Theorem 4. *As $\delta c_3 m_1 > 0$, the second order approximation of a homoclinic orbit for the Liénard system* (3.1)*, close to the bifurcation point, is given by:*

$$\begin{aligned} x_1(s) = {} & \frac{\varepsilon\,\mathrm{sech}^3(s)}{75\sqrt{2}c_3^{5/2}}\{75c_3^2 + \cosh(2s)[75c_3^2 - \delta^2\varepsilon^2 m_1^2(7 + 12\ln\cosh(s) - 6\ln^2\cosh(s))] \\ & + 3\delta^2\varepsilon^2 m_1^2[3 + 4\ln\cosh(s) - 6\ln^2\cosh(s)] + 6\delta\varepsilon m_1 \sinh(2s)[\delta\varepsilon m_1 s + 5c_3\ln\cosh(s)]\} \\ x_2(s) = {} & -\frac{\varepsilon^2\,\mathrm{sech}^4(s)}{75\sqrt{2}c_3^{5/2}}\{3\delta\varepsilon m_1\cosh(3s)[-5c_3 + \delta\varepsilon m_1 s + 5c_3\ln\cosh(s)] \\ & - 15\delta\varepsilon m_1\cosh(s)[-c_3 + \delta\varepsilon m_1 s + 5c_3\ln\cosh(s)] \\ & + \sinh(s)[75c_3^2 + \delta^2\varepsilon^2(23m_1^2 + 300c_3 m_0 m_2 + 96m_1^2\ln\cosh(s) \\ & - 66m_1^2\ln^2\cosh(s)) + \cosh(2s)[75c_3^2 - \delta^2\varepsilon^2 m_1^2(1 + 24\ln\cosh(s) - 6\ln^2\cosh(s))]]\}, \end{aligned}$$

where $\varepsilon = \sqrt{-c1}$ and m_0 is given by (3.4).

Replacing u_i, v_i, $i = 0,1,2$ given by (9), (11), (12) from [15], into (3.6), and using (3.2), with $\sigma = 1$, the following result is obtained.

Theorem 5. *As $\delta c_3 m_1 < 0$, the second order approximation of a heteroclinic orbit for the Liénard system* (3.1)*, close to the bifurcation point, is given by:*

$$x_1(s) = \frac{\varepsilon}{\sqrt{-c_3}}\tanh\frac{s}{\sqrt{2}} - \frac{\varepsilon^2\delta m_1 \operatorname{sech}^2\frac{s}{\sqrt{2}}}{150c_3^2\sqrt{-c_3}}\left[3\sqrt{2}\left(\delta\varepsilon m_1 s + 10c_3\ln\cosh\frac{s}{\sqrt{2}}\right)\right.$$
$$\left. +4\delta\varepsilon m_1\tanh\frac{s}{\sqrt{2}}\left(-2-3\ln\cosh\frac{s}{\sqrt{2}}+3\ln^2\cosh\frac{s}{\sqrt{2}}\right)\right],$$
$$x_2(s) = \frac{\varepsilon^2 \operatorname{sec}h^4\frac{s}{\sqrt{2}}}{300(-c_3)^{5/2}}\left\{75\sqrt{2}c_3(c_3+\delta^2\varepsilon^2 m_0 m_2)+\sqrt{2}\delta^2\varepsilon^2 m_1^2\left(7+36\ln\cosh\frac{s}{\sqrt{2}}-24\ln^2\cosh\frac{s}{\sqrt{2}}\right)\right.$$
$$+\sqrt{2}\cosh(s\sqrt{2})\left[75c_3(c_3-\delta^2\varepsilon^2 m_0 m_2)-\delta^2\varepsilon^2 m_1^2\left(5+24\ln\cosh\frac{s}{\sqrt{2}}-12\ln^2\cosh\frac{s}{\sqrt{2}}\right)\right]$$
$$\left.+6\delta\varepsilon m_1\sinh(s\sqrt{2})\left[-5c_3+\delta\varepsilon m_1 s+10c_3\ln\cosh\frac{s}{\sqrt{2}}\right]\right\}$$

with $\varepsilon = \sqrt{c_1}$ and m_0 given by (3.5)*.*

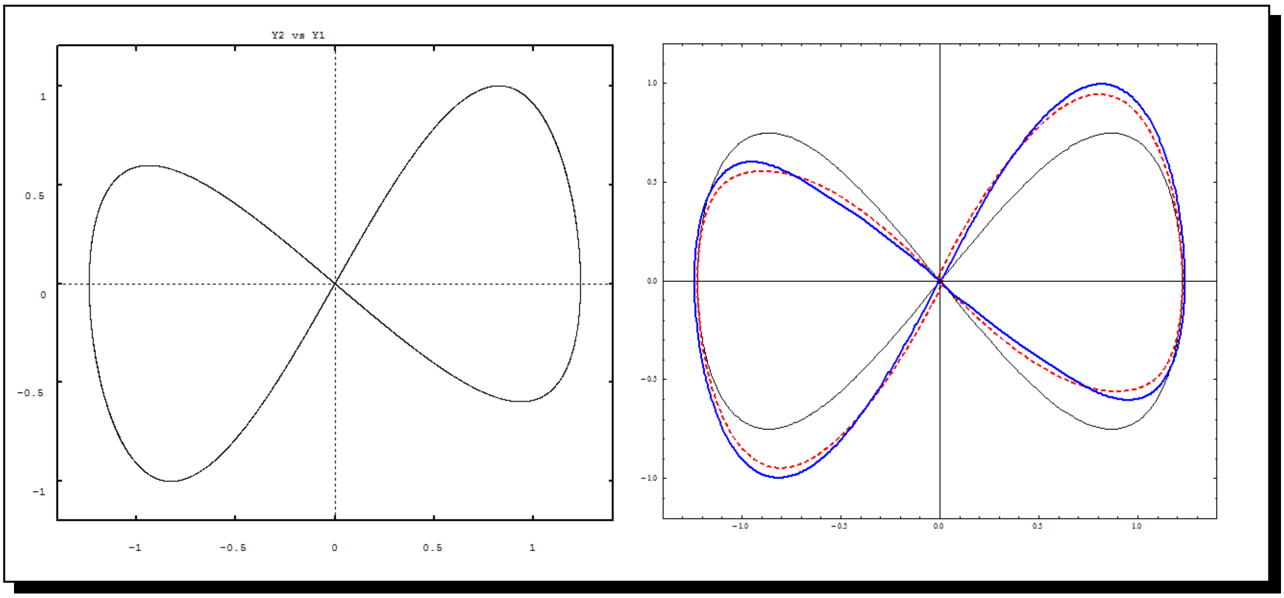

Figure 2. Homoclinic orbits for Liénard system (3.1). Parameters $c_1 = -1.5$, $c_3 = 2$, $m_1 = 1$, $m_2 = 0$, and $\delta = 1.5$: (i) numerical approximation (to the left); (ii) Hamiltonian (in black), first (red, dotted) and second order (blue) approximations (to the right).

As an example, consider $c_1 = -1.5$, $c_3 = 2$, $m_1 = 1$, $m_2 = 0$, and $\delta = 1.5$. Using the first order approximation of the curve of homoclinic bifurcation values, we obtain $m_0 = 0.6$, while using the second order approximation (3.4) we obtain $m_0 = 0.6088$. This last value is a better approximation of the numerically value $m_0 = 0.60875$ (obtained using XPPAUT [7]), than the first order one. The homoclinic orbits in this case are plotted in Figure 2, using [16]. A neighbourhood of origin in Figure 2(ii) is represented in Figure 3. Remark that the curve of first order approximation of the homoclinic orbit makes a parasitic turn before approaching the saddle equilibrium in origin as the time tends to $-\infty$. This phenomenon does not happen for the curve of second order approximation of this homoclinic.

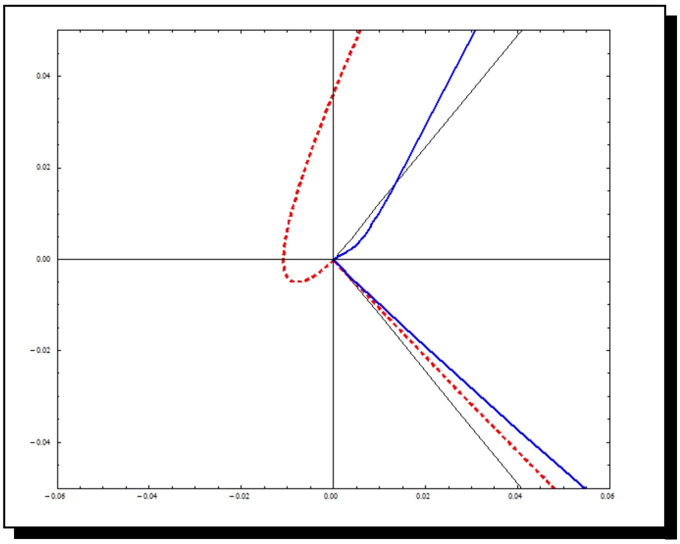

Figure 3. Detail in Figure 2(ii).

4. Conclusions

In our study, we found the second order approximations of homoclinics involved in the normal form of a degenerated Bogdanov-Takens bifurcation, truncated up to five order terms and with parameter dependent coefficients. They are given in Theorem 1. In addition, the second order approximation of the curve of homoclinic bifurcation values was obtained in Theorem 2. These formulae and some similar ones from a previous paper concerning the heteroclinic orbits, were applied for a Liénard system in Section 3. Remark that the second order approximation for the value corresponding to the homoclinic bifurcation given by (3.4) or corresponding to the heteroclinic bifurcation given by (3.5), is good also far away from the bifurcation point, even though the second order approximation of the homoclinic/heteroclinic orbit fails to be in good agreement with the numerical one.

Acknowledgement

This research was partially supported by Horizon2020-2017-RISE-777911 project.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] B. Al-Hdaibat, W. Govaerts, Y.A. Kuznetsov and H.G.E. Meijer, Initialization of Homoclinic solutions near Bogdanov–Takens Points: Lindstedt–Poincaré compared with regular perturbation method, *SIAM J. Appl. Dynam. Sys.* **15** (2016), 952, DOI: 10.1137/15M1017491.

[2] M. Belhaq, F. Lakrad abd A. Fahsi, Predicting Homoclinic bifurcations in planar autonomous systems, *Nonlinear Dynam.* **18** (1999), 303, DOI: 10.1023/A:1026428718802.

[3] M. Belhaq and F. Lakrad, Prediction of homoclinic bifurcation: the elliptic averaging method, *Chaos, Solitons & Fractals* **11** (2000), 2251, DOI: 10.1016/S0960-0779(99)00144-7.

[4] Y.Y. Cao, K.W. Chung and J. Xu, A novel construction of homoclinic and heteroclinic orbits in nonlinear oscillators by a perturbation-incremental method, *Nonlinear Dynam.* **64** (2011), 221, DOI: 10.1007/s11071-011-9990-9.

[5] Y.Y. Chen and S.H. Chen, Homoclinic and heteroclinic solutions of cubic strongly nonlinear autonomous oscillators by the hyperbolic perturbation method, *Nonlinear Dynam.* **58** (2009), 417, DOI: 10.1007/s11071-009-9489-9.

[6] S.N. Chow, C. Li and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Chapter 4 Cambridge University Press, Cambridge and New York (1994).

[7] B. Ermentrout, *Simulating, Analyzing, and Animating Dynamical Systems: A Guide to XPPAUT for Researchers and Students*, SIAM, Philadelphia (2002).

[8] B. Feng and R. Hu, A survey on homoclinic and heteroclinic orbits, *Applied Mathematics E-Notes* **2** (2003), 16.

[9] J. Feng, Q. Zhang, W. Wang and S.Hao, A new approach of asymmetric homoclinic and heteroclinic orbits construction in several typical systems based on the undetermined Padé approximation method, *Mathematical Problems in Engineering* **2016** (2016), 8585290, DOI: 10.1155/2016/8585290.

[10] A. Gasull, H. Giacomini and J. Torregrosa, Some results on homoclinic and heteroclinic connections in planar systems, *Nonlinearity* **23** (2010), 2977, DOI: 10.1088/0951-7715/23/12/001.

[11] A.I. Khibnik, B. Krauskopf and C. Rousseau, Global study of a family of cubic Liénard equations, *Nonlinearity* **11** (1998), 1, DOI: 10.1088/0951-7715/11/6/005.

[12] Y.A. Kuznetsov, H.G.E. Meijer, B. Al-Hdaibat and W. Govaerts, Improved homoclinic predictor for Bogdanov-Takens bifurcation, *International Journal of Bifurcation and Chaos* **24** (2014), 1450057, DOI: 10.1142/S0218127414500576.

[13] Y.A. Kuznetsov, H.G.E. Meijer, B. Al-Hdaibat and W. Govaerts, Accurate approximation of Homoclinic solutions in Gray-Scott kinetic model, *International Journal of Bifurcation and Chaos* **25** (2015), 1450057, DOI: 10.1142/S0218127415501254.

[14] C. Rocşoreanu and M. Sterpu, Approximations of the homoclinic orbits near a double-zero bifurcation with symmetry of order two, *International Journal of Bifurcation and Chaos* **27** (2017), 1750109, DOI: 10.1142/S0218127417501097.

[15] C. Rocşoreanu and M. Sterpu, Approximations of the heteroclinic orbits near a double-zero bifurcation with symmetry of order two. Application to a Liénard equation, *International Journal of Bifurcation and Chaos*, communicated (2018).

[16] Wolfram Research, Inc., *Mathematica*, Version 8.0, Champaign, IL (2010).

[17] Q.-C. Zhang, W. Wang and W.-Y. Li, Heteroclinic bifurcation of strongly nonlinear oscillator, *Chinese Physics Letters* **25** (2008), 1905s.