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Research Article

Determination of Interlayer Stresses for Multilayered Materials by using Biquadratic Basic Functions

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Abstract. In this paper, we deal with the boundary value problem for the Lame system, modeling the contact problem for a multilayered material. We use biquadratic basic functions to obtain the transmission conditions on the boundaries of interlayer by the Finite Element Method. We analyze the interlayer stresses.

Keywords. Finite-element analysis; Multilayered Material; Transmission conditions

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1. Introduction

The contact problem related to the deformation of a rigid punch attracts the interest of mathematicians, mechanics and engineers for many years. The problem was considered by many authors: a historical review one can find in [7], [11], [16], [17] and [22]. Nowadays, the problem has not lost its relevance (see the series of articles by Aleksandrov and his coauthors [1]–[6], by Borodich and his coauthors [8]–[12] and by Komvopoulos and his coauthors [16], [18], [19],

[23]. We would like to point out some works here. The paper [7] is devoted to the analysis of the infinitesimal deformations of a linear elastic anisotropic layer by using Stroh formalism method. The work [22] deals with the contact problem of a stiff spherical indenter with a composite plate by dint of the commercial software and the problem are simulated by a 2-D axisymmetric model. The results numerically obtained in [22] show independence of the indentation response of an orthotropic laminate from the material, the authors demonstrate dependence of the thickness of the multilayered material. In the paper [15] plane and axisymmetric contact problems for a three-layered elastic half-space are consided. A plane problem reduces to a singular integral equation with a Cauchy kernel in [15]. An analytical solution of this type of equations one can find in [20]. In turn, to obtain the solution of the integral equation the method of reduction to the corresponding conjugation problem can be used [21]. In addition, the solution of the integral equation can be obtained by numerical methods.

In the present paper, we give an analysis and numerical solution of the boundary value problem for the Lame system, modeling the contact problem for a multilayered material. By using the biquadratic basic functions, the transmission conditions are obtained on the boundaries of interlayer by the Finite Element Method and the interlayer stresses are analyzed.

2. Statement of the Problem

The mathematical model of the contact problem related to the deformation of a rigid punch with a frictional pressure of a finite dimensional elastic material, which is a quadrilateral region, is expressed by the boundary value problem for the Lame system as follows [13]:

$$\begin{cases} (\lambda+2\mu)\dfrac{\partial^2 u_1(x_1,x_2)}{\partial x_1^2}+\mu\dfrac{\partial^2 u_1(x_1,x_2)}{\partial x_2^2}+(\lambda+\mu)\dfrac{\partial^2 u_2(x_1,x_2)}{\partial x_1\partial x_2}+F_1=0, \\ \mu\dfrac{\partial^2 u_2(x_1,x_2)}{\partial x_1^2}+(\lambda+2\mu)\dfrac{\partial^2 u_2(x_1,x_2)}{\partial x_2^2}+(\lambda+\mu)\dfrac{\partial^2 u_1(x_1,x_2)}{\partial x_1\partial x_2}+F_2=0, \quad (x_1,x_2)\in\Omega; \end{cases} \tag{2.1}$$

$$\begin{cases} u_2(x_1,0)\le -\alpha+\varphi(x_1), \quad \sigma_{22}(u(x_1,x_2))\le 0, & \\ \sigma_{22}(u(x_1,x_2))\left[u_2(x_1,0)+\alpha-\varphi(x_1)\right]=0, & (x_1,x_2)\in\Gamma_0; \\ \sigma_{11}(u(x_1,x_2))=0, & (x_1,x_2)\in\Gamma_\sigma; \\ u_1(x_1,x_2)=0, \quad u_2(x_1,x_2)=0, & (x_1,x_2)\in\Gamma_u; \\ u_1(0,x_2)=0, & (0,x_2)\in\Gamma_1; \\ \sigma_{12}(u(x_1,x_2))=0, & (x_1,x_2)\in\partial\Omega. \end{cases} \tag{2.2}$$

Here $\Omega := \{(x_1,x_2)\in R^2 : 0<x_1<l_{x_1}, -l_{x_2}<x_2<0, l_{x_1}>0, l_{x_2}>0\}$ is the region occupied by the cross-section of the material under the influence of the punch and $\Gamma_0,\Gamma_\sigma,\Gamma_u,\Gamma_1\subset\partial\Omega$ are the relevant parts of the boundary of the region Ω (Figure 1). Namely, $\Gamma_\sigma=\{(l_{x_1},x_2): -l_{x_2}<x_2<0\}$, $\Gamma_u=\{(x_1,-l_{x_2}): 0<x_1<l_{x_1}\}$, $\Gamma_0=\{(x_1,0): 0<x_1<l_{x_1}\}$, $\Gamma_1=\{(0,x_2): -l_{x_2}<x_2<0\}$. Since the condition on the upper boundary Γ_0 is given by inequality, the contact region of the punch $\Gamma_c=\{(x_1,x_2)\in\Gamma_0 : u_2=-\alpha+\varphi(x_1)\}$ is not certain and the problem is nonlinear.

In this work we study the problem of the deformation for the multilayered material. The layers of the material formed by different materials with Lame constants λ_k,μ_k of

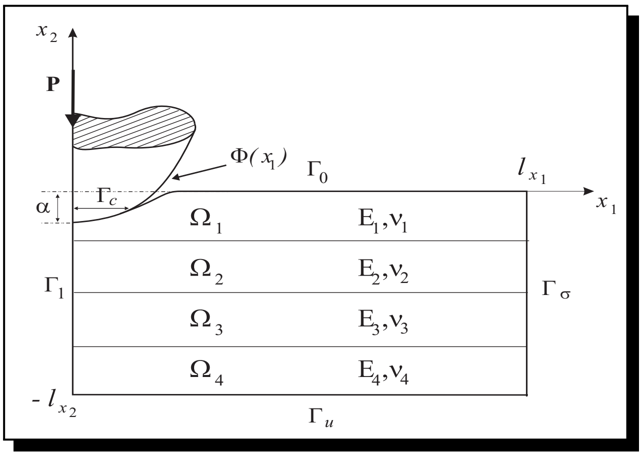

Figure 1. Geometry of the spherical indentation

the Ω_k layers forming the Ω region; $u=(u_1(x_1,x_2),u_2(x_1,x_2))$ is displacement function and $\sigma_{ii}(u)=\lambda_k \div(u)+2\mu_k \partial u_i/\partial x_i$, $\sigma_{ij}(u)=\mu_k\left(\partial u_i/\partial x_j+\partial u_j/\partial x_i\right)$, $i,j\in\{1,2\}$, are components of stress tensors; $\varphi(x_1)$ is the surface of the indentor; α is the maximum value of the indentation depth; $F=(F_1,F_2)$ is the internal force vector.

Let us find a variation statement of the problem (2.1)-(2.2). Let us multiply the first and the second equations of (2.1) by v_1-u_1 and v_2-u_2, respectively. Then, let us sum up the results of multiplying and integrate on $\Omega=\bigcup_k \Omega_k$. We have

$$-\int_\Omega\int\left\{\frac{\partial\sigma_{11}(u)}{\partial x_1}(v_1-u_1)+\frac{\partial\sigma_{12}(u)}{\partial x_2}(v_1-u_1)+\frac{\partial\sigma_{21}(u)}{\partial x_1}(v_2-u_2)+\frac{\partial\sigma_{22}(u)}{\partial x_1}(v_2-u_2)\right\}dx_1dx_2$$
$$=\int_\Omega\int\{F_1(v_1-u_1)-F_2(v_2-u_2)\}dx_1dx_2.$$

Using well-known variational technique (see [17]), we can find the solution of unilateral contact problem (2.1)-(2.2). The solution of the problem (2.1)-(2.2) minimizes by the following functional

$$J(u)=(Au,u)-0.5b(v),\quad u\in V,$$

on the set

$$V=\{u\in H^1(\Omega):u_1(0,x_2)=0,\ (0,x_2)\in\Gamma_1;\ u_1(x_1,-l_{x_2})=u_2(x_1,-l_{x_2})=0,$$
$$(x_1,-l_{x_2})\in\Gamma_u;\ u_2(x_1,0)\le-\alpha+\varphi(x_1),\ (x_1,0)\in\Gamma_0\}$$

in the Sobolev space $H^1(\Omega):=W_2^1(\Omega)\times W_2^1(\Omega)$.

Here the bilinear and the linear parts of above functional have the following form

$$(Au,v)=\sum_k\int\int_{\Omega_k}\left\{\left[(\lambda_k+2\mu_k)\frac{\partial u_1}{\partial x_1}+\lambda_k\frac{\partial u_2}{\partial x_2}\right]\frac{\partial v_1}{\partial x_1}+\mu_k\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right)\right.$$
$$\left.+\left[\lambda_k\frac{\partial u_1}{\partial x_1}+(\lambda_k+2\mu_k)\frac{\partial u_2}{\partial x_2}\right]\frac{\partial v_2}{\partial x_2}\right\}dx_1dx_2, \tag{2.3}$$

$$b(v)=\sum_k\int\int_{\Omega_k}[F_1v_1+F_2v_2]dx_1dx_2, \tag{2.4}$$

respectively.

3. Finite-Element Formulation

Let us use here the biquadratic basic functions $\xi_{ij}(x_1,x_2)$ to analyze the problem (2.1)-(2.2). Here $\xi_{ij}(x_{1,pq},x_{2,pq}) = \begin{cases} 1 & (i,j)=(p,q), \\ 0 & (i,j)\neq(p,q) \end{cases}$ and the compact support of the basic function is $\overline{\Omega}_{ij} = \overline{e_{i-1j-1}\cup e_{i-1j}\cup e_{ij-1}\cup e_{ij}}$. A numerical solution (u^h,v^h) has the following form:

$$u^h(x_1,x_2)=\sum_{(ij)} u_{ij}\xi_{ij}(x_1,x_2), \quad v^h(x_1,x_2)=\sum_{(ij)} v_{ij}\xi_{ij}(x_1,x_2).$$

The *local stiffness matrix* (LSM) of the finite element e_{ij} is constructed as follows

$$\mathcal{L}_{ij}=\begin{bmatrix} \mathcal{L}_{11}^{(ij)} & \mathcal{L}_{12}^{(ij)} \\ \mathcal{L}_{21}^{(ij)} & \mathcal{L}_{22}^{(ij)} \end{bmatrix}.$$

Elements of the LSM are calculated by the formulas

$$\left[\mathcal{L}_{11}^{(ij)}\right]=\left[\iint_{e_{ij}}\left\{(\lambda+2\mu)\frac{\partial\xi_{ij}}{\partial x_1}\frac{\partial\xi_{kl}}{\partial x_1}+\mu\frac{\partial\xi_{ij}}{\partial x_2}\frac{\partial\xi_{kl}}{\partial x_2}\right\}dx_1dx_2\right],$$

$$\left[\mathcal{L}_{12}^{(ij)}\right]=\left[\iint_{e_{ij}}\left\{\lambda\frac{\partial\xi_{ij}}{\partial x_2}\frac{\partial\xi_{kl}}{\partial x_1}+\mu\frac{\partial\xi_{ij}}{\partial x_1}\frac{\partial\xi_{kl}}{\partial x_2}\right\}dx_1dx_2\right],$$

$$\left[\mathcal{L}_{21}^{(ij)}\right]=\left[\iint_{e_{ij}}\left\{\lambda\frac{\partial\xi_{ij}}{\partial x_1}\frac{\partial\xi_{kl}}{\partial x_2}+\mu\frac{\partial\xi_{ij}}{\partial x_2}\frac{\partial\xi_{kl}}{\partial x_1}\right\}dx_1dx_2\right],$$

$$\left[\mathcal{L}_{22}^{(ij)}\right]=\left[\iint_{e_{ij}}\left\{(\lambda+2\mu)\frac{\partial\xi_{ij}}{\partial x_2}\frac{\partial\xi_{kl}}{\partial x_2}+\mu\frac{\partial\xi_{ij}}{\partial x_1}\frac{\partial\xi_{kl}}{\partial x_1}\right\}dx_1dx_2\right], \quad k,l=\overline{1,9}.$$

So, using well-known finite-element technology we calculate the LSM $\mathcal{L}_{ij}=\{(l_{pq})\}$, $p,q=\overline{1,18}$ for the elements e_{ij}. We can define unknown vectors corresponding to e_{i-1j-1}, e_{i-1j}, e_{ij-1}, e_{ij} neighboring with point (i,j) (Figure 2) as follows:

$$\begin{aligned}
\omega_{i-1j-1}=(&u_{i-1j-1},u_{i-1j-\frac12},u_{i-1j},u_{i-\frac12j-1},u_{i-\frac12j-\frac12},u_{i-\frac12j},u_{ij-1},u_{ij-\frac12},u_{ij},\\
&v_{i-1j-1},v_{i-1j-\frac12},v_{i-1j},v_{i-\frac12j-1},v_{i-\frac12j-\frac12},v_{i-\frac12j},v_{ij-1},v_{ij-\frac12},v_{ij}),\\
\omega_{i-1j}=(&u_{i-1j},u_{i-1j+\frac12},u_{i-1j+1},u_{i-\frac12j},u_{i-\frac12j+\frac12},u_{i-\frac12j+1},u_{ij},u_{ij+\frac12},u_{ij+1},\\
&v_{i-1j},v_{i-1j+\frac12},v_{i-1j+1},v_{i-\frac12j},v_{i-\frac12j+\frac12},v_{i-\frac12j+1},v_{ij},v_{ij+\frac12},v_{ij+1}),\\
\omega_{ij-1}=(&u_{ij-1},u_{ij-\frac12},u_{ij},u_{i+\frac12j-1},u_{i+\frac12j-\frac12},u_{i+\frac12j},u_{i+1j-1},u_{i+1j-\frac12},u_{i+1j},\\
&v_{ij-1},v_{ij-\frac12},v_{ij},v_{i+\frac12j-1},v_{i+\frac12j-\frac12},v_{i+\frac12j},v_{i+1j-1},v_{i+1j-\frac12},v_{i+1j}),\\
\omega_{ij}=(&u_{ij},u_{ij+\frac12},u_{ij+1},u_{i+\frac12j},u_{i+\frac12j+\frac12},u_{i+\frac12j+1},u_{i+1j},u_{i+1j+\frac12},u_{i+1j+1},\\
&v_{ij},v_{ij+\frac12},v_{ij+1},v_{i+\frac12j},v_{i+\frac12j+\frac12},v_{i+\frac12j+1},v_{i+1j},v_{i+1j+\frac12},v_{i+1j+1}).
\end{aligned}$$

The nodal points of all finite elements are numerated from down to up and from left to right. The finite-element e_{ij} has its index (ij) corresponding to the lower-left vertice.

In this context, to derive the discrete analogue of equilibrium equation, as well as contact and interlaminar stresses, the following technique is suggested.

In order to obtain the equation for the central point of the finite element we have to multiply the displacement vector corresponding to this finite element with ninth (tenth) line of the LSM.

In order to obtain the discrete form for the Lame system (3.1) on the nodal points of mesh $(x_{1,ij}, x_{2,ij}) \in \Omega_k$ we use four finite elements neighbouring with this point (Figure 2). So, using the components of the local stiffness matrix and above four vectors we can write their contribution to the discrete form of first (*second*) equilibrium equation in the form

$$\begin{aligned}&(\text{seventeenth } (\textit{eighteenth}) \text{ line of } \mathcal{L}_{i-1j-1}) \times \omega^T_{i-1j-1}\\ &\quad + (\text{thirteenth } (\textit{fourteenth} \text{ line of } \mathcal{L}_{i-1j}) \times \omega^T_{i-1j}\\ &\quad + (\text{fifth } (\textit{sixth}) \text{ line of } \mathcal{L}_{ij-1}) \times \omega^T_{ij-1}\\ &\quad + (\text{first } (\textit{second}) \text{ line of } \mathcal{L}_{ij}) \times \omega^T_{ij}.\end{aligned} \tag{3.1}$$

After non difficult transformations, we can write the discrete form for the system (1) on k-th layer Ω_k by using finite differences notations:

$$\begin{cases} -h_i\tau_j\left[(\lambda_k+2\mu_k)u_{\overline{x_1}x_1}+\mu_k u_{\overline{x_2}x_2}+\dfrac{\lambda_k+\mu_k}{2}(v_{x_1x_2}+v_{\overline{x_1}\,\overline{x_2}})\right]=F^h_{1,ij}, \\ -h_i\tau_j\left[\mu_k v_{\overline{x_1}x_1}+(\lambda_k+2\mu_k)v_{\overline{x_2}x_2}+\dfrac{\lambda_k+\mu_k}{2}(u_{x_1x_2}+u_{\overline{x_1}\,\overline{x_2}})\right]=F^h_{2,ij}, \end{cases} \tag{3.2}$$

where $F^h_{1,ij}$ and $F^h_{2,ij}$ are values of components of internal forces F on the nodal point $(x_{1,ij}, x_{2,ij})$. by using the notations of finite differences

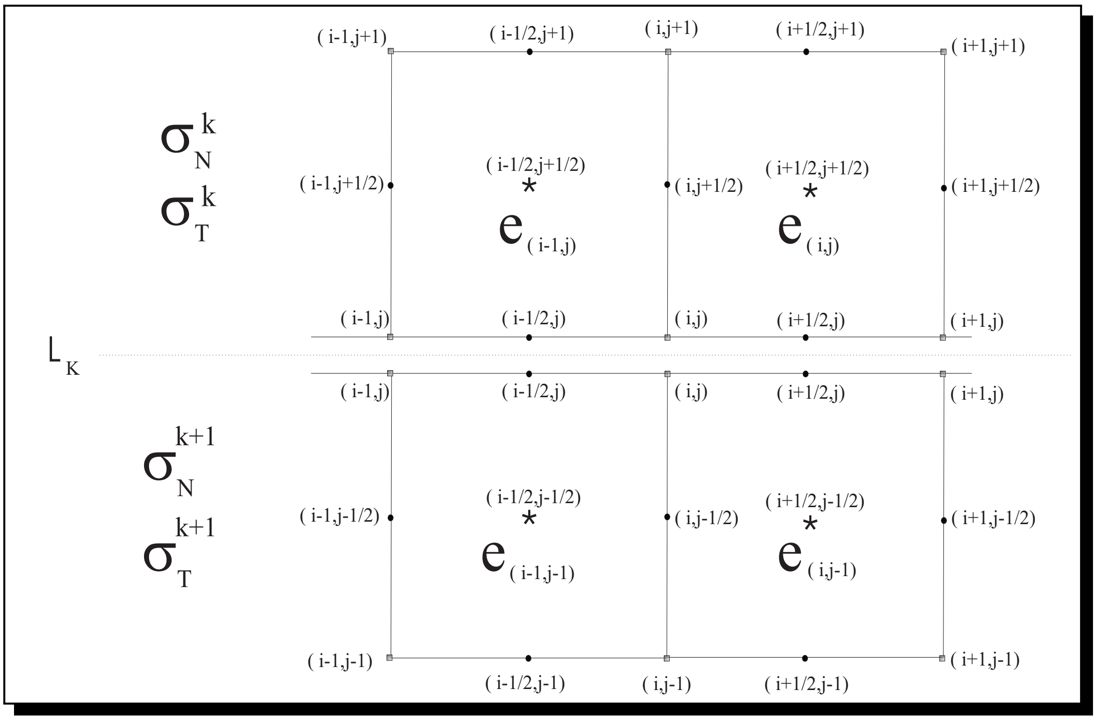

Figure 2. The interlayer finite elements

Let us denote interlayer stress on the common border by $\sigma^k_{N,ij}$ ($\sigma^k_{T,ij}$), $\sigma^{k+1}_{N,ij}$ ($\sigma^{k+1}_{T,ij}$). In order to obtain the approximating expression of $\sigma^k_{N,ij}$ ($\sigma^k_{T,ij}$) we have to multiply $\mathcal{L}_{i-1j}$ by ω_{i-1j} and $\mathcal{L}_{ij}$ by ω_{ij}, respectively. Then we have to sum up the results of that multiplying. Now let us approximate $\sigma^{k+1}_{N,ij}$ ($\sigma^{k+1}_{T,ij}$) on the upper boundary of lower layer. In order to do that we have to multiply line eighteenth (seventeenth) of $\mathcal{L}_{i-1j-1}$ by ω_{i-1j-1} and line sixth (fifth) of $\mathcal{L}_{ij-1}$ by ω_{ij-1}, respectively. Then, we have to sum up the results of that multiplying.

The discrete analogues of normal ($\sigma^{h,(k)}_N$) and tangential ($\sigma^{h,(k)}_T$) components of stresses on k-th layer Ω_k have the following form:

$$\sigma^{h,(k)}_N = -\lambda_k \frac{u_{\overline{x_1}} + \check{u}_{x_1}}{2} - (\lambda_k + 2\mu_k) v_{\overline{x_2}} - \frac{\tau}{2}\mu_k [u_{x_1 x_2} + v_{\overline{x_1} x_1}], \tag{3.3}$$

$$\sigma^{h,(k)}_T = -\mu_k (u_{x_2} + \frac{v_{\overline{x_1}} + \hat{v}_{x_1}}{2}) - \frac{\tau}{2}[(\lambda_k + 2\mu_k) u_{\overline{x_1} x_1} + \lambda_k v_{x_1 x_2}]. \tag{3.4}$$

Analogously, we can obtain $\sigma^{h,(k+1)}_N$ and $\sigma^{h,(k+1)}_T$ as follows

$$\sigma^{h,(k+1)}_N = \lambda_{k+1} \frac{u_{x_1} + \check{u}_{\overline{x_1}}}{2} + (\lambda_{k+1} + 2\mu_{k+1}) v_{\overline{x_2}} - \frac{\tau}{2}\mu_{k+1} (u_{\overline{x_1}\,\overline{x_2}} + v_{\overline{x_1} x_1}), \tag{3.5}$$

$$\sigma^{h,(k+1)}_T = \mu_{k+1} (u_{\overline{x_2}} + \frac{v_{x_1} + \check{v}_{\overline{x_1}}}{2}) - \frac{\tau}{2}[(\lambda_{k+1} + 2\mu_{k+1}) u_{\overline{x_1} x_1} + \lambda_{k+1} v_{\overline{x_1}\,\overline{x_2}}]. \tag{3.6}$$

Using (3.3)-(3.6), it is not difficult to show that the following transmission conditions

$$\sigma^{h,(k)}_T - \sigma^{h,(k+1)}_T = 0, \quad \sigma^{h,(k)}_N - \sigma^{h,(k+1)}_N = 0 \tag{3.7}$$

are satisfied. In order to determinate contact domain a_c we have to calculate σ_N on the upper side of the body. We calculate this value the same way as for the upper boundary of lower layer.

4. Numerical Realization of Method

In order to illustrate the method let us consider the following domain

$$\Omega = \{(x_1, x_2) \mid 0 \le x_1 \le 1.5, -1 \le x_2 \le 0\}$$

and consider the size mesh $N_{x_1} \times N_{x_2} = 50 \times 21$ in the rectangular region Ω. In order to carry out numerical experiments let us consider two examples for two layers materials: iron-copper and iron-steel. Let us refresh, that copper and steel more soft than iron. The upper layer in both examples is iron. The elasticity modules and Poisson's constants of these materials are $E_{Fe} = 30000$ [kN/cm^2], $\nu_{Fe} = 0.27$, $E_{Cu} = 18100$ [kN/cm^2], $\nu_{Cu} = 0.36$, $E_{St} = 21000$ [kN/cm^2], $\nu_{St} = 0.3$.

In order to clarify the contact domain a_c we use the multigrid method ([14]). On the p-th step, $a_c = \partial\Gamma_{c,h}$ has to be satisfied one of the following conditions:

$$0 < a^{(p)}_c < a_c < a^{(p+1)}_c < \ell_{x_1} \quad (\text{or } 0 < a^{(p+1)}_c < a_c < a^{(p)}_c < \ell_{x_1}).$$

Here $a^{(p)}_c$ and $a^{(p+1)}_c$ are corresponding to the p-th and $(p+1)$-th iterations, respectively.

The following inequalities have to be satisfied on the mesh points $a_c^{(p)}$ and $a_c^{(p+1)}$:

$$\sigma_{N,h}(a_c^{(p)}) < 0, \quad \sigma_{N,h}(a_c^{(p+1)}) > 0. \tag{4.1}$$

Usıng

$$\gamma = \frac{\sigma_{N,h}(a_c^{(p+1)})}{\sigma_{N,h}(a_c^{(p)}) + \sigma_{N,h}(a_c^{(p+1)})} \tag{4.2}$$

we obtain

$$a_c = \gamma a_c^{(p)} + (1-\gamma) a_c^{(p+1)}, \quad \gamma \in (0,1). \tag{4.3}$$

The process of iteration is repeated until satisfaction the next inequality:

$$|\sigma_{N,h}(a_c)| < \epsilon. \tag{4.4}$$

Here $\epsilon > 0$ is a given accuracy. For example, it takes about five-six iterations to clarify a_c for $\epsilon = 10^{-3}$.

For the thickness of the iron layer 0.1, 0.2, 0.4, 0.5 respectively, we have Figure 3 and Figure 4. In these figures one can find normal and tangential components of the interlayer stresses and the stresses on the upper part of the body.

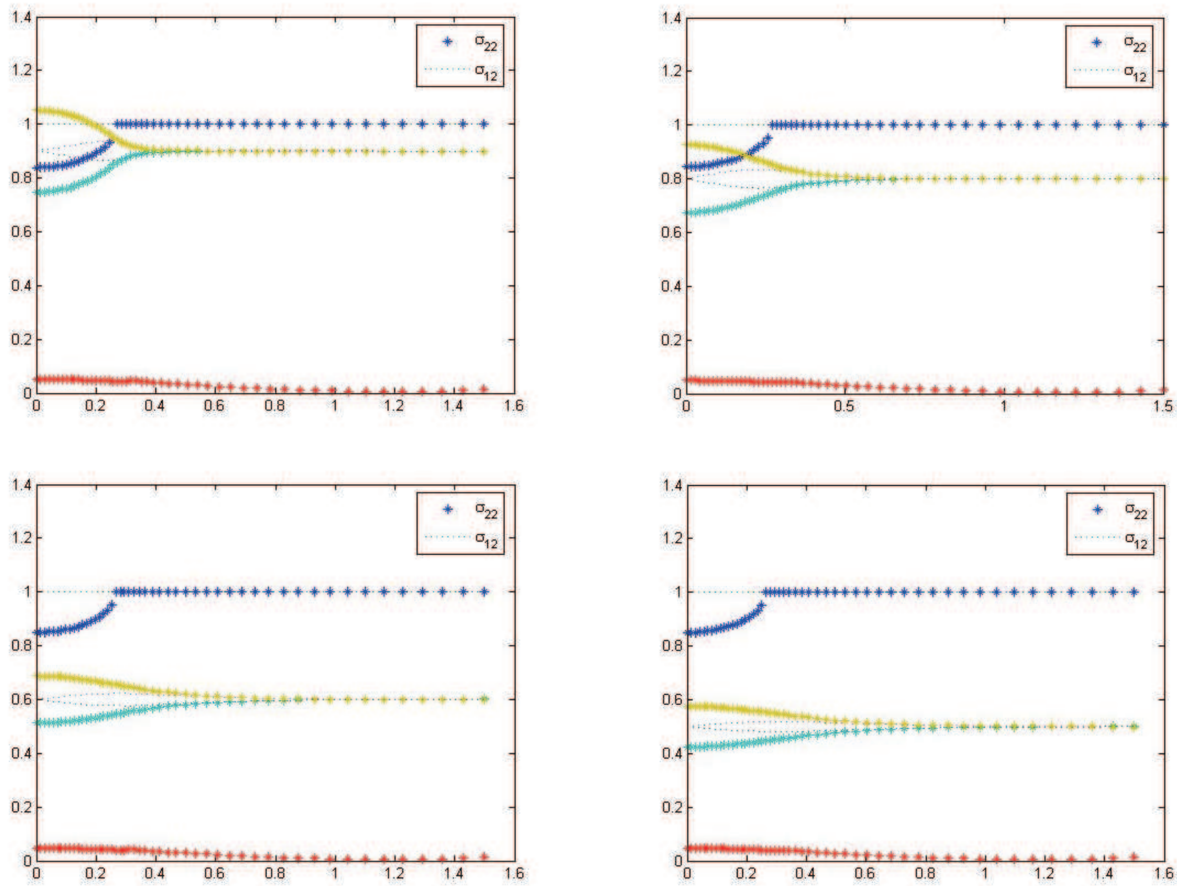

Figure 3. Interlayer stress graphics for multilayered material composed of iron and copper

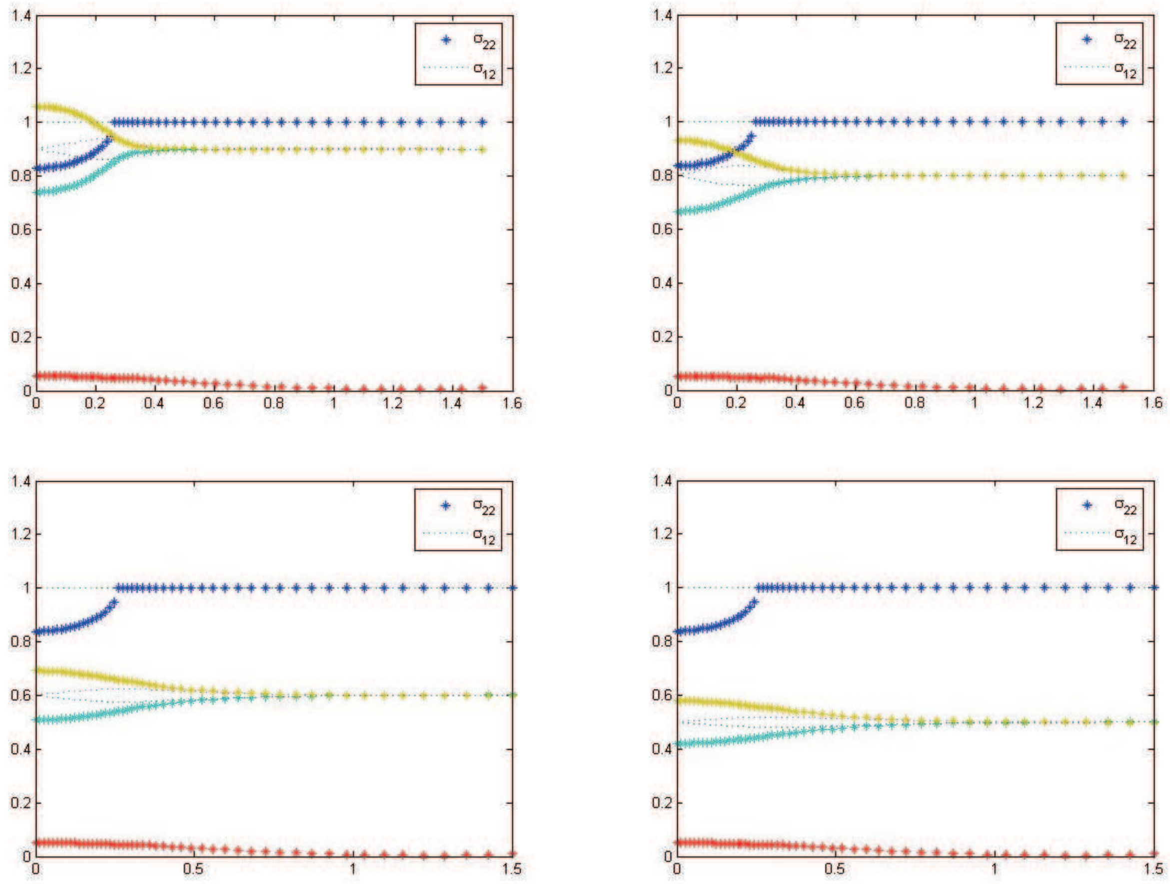

Figure 4. Interlayer stress graphics for multilayered material composed of iron and steel

For the value of the indentation depth $\alpha = 0.035$ cm and for the different thickness of layers we obtain the contact domain a_c and the values of force (P) effected the body, calculated by the formula $P = -\int_{\Gamma_c(\alpha)} \sigma_{N,h}(x_1)dx_1$ (see Table 1).

Table 1. The obtained values P and a_c corresponding to the different thickness h_{Fe}

	$E_{Fe} = 30000$ [kN/cm^2], $\nu_{Fe} = 0.27$ $E_{Cu} = 18100$ [kN/cm^2], $\nu_{Cu} = 0.36$		$E_{Fe} = 30000$ [kN/cm^2], $\nu_{Fe} = 0.27$ $E_{St} = 21000$ [kN/cm^2], $\nu_{St} = 0.3$	
h_{Fe}[cm]	$P \times 10^2$	a_c	$P \times 10^2$	a_c
0.1	3.4052	0.2720	3.5009	0.2649
0.2	3.2611	0.2704	3.3830	0.2637
0.3	3.2473	0.2689	3.3301	0.2624
0.4	3.1462	0.2667	3.2901	0.2604
0.5	3.1058	0.2642	3.2587	0.2581

The relation between the thickness of the upper layer and P and between the thickness of the upper layer and a_c are given in Figure 5.

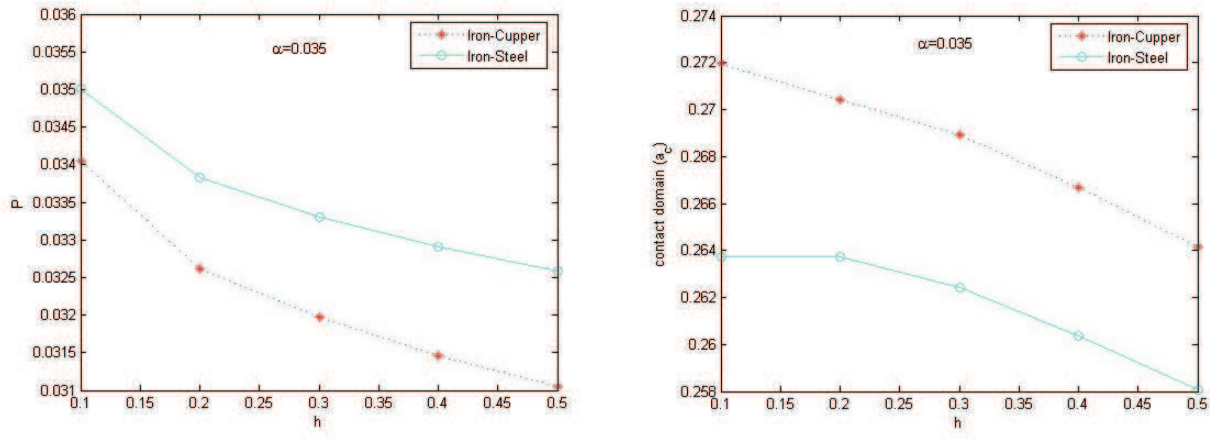

Figure 5. The graphics $P(h)$ and $a_c(h)$ for iron-copper and iron-steel body

5. Conclusion

In this paper we study boundary value problem for the Lame system, modeling the contact problem for a multilayered material. By using the biquadratic basic functions, the transmission conditions are obtained on the boundaries of interlayer by the Finite Element Method and the interlayer stresses are analyzed. The results obtained is the mathematical backgrounds for calculating the mechanical and geometric characteristics of a laminate. So, our research can be a mathematical background for the study of the inverse problems.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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