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Existence of Coincidence and Fixed Point Theorems for Non-linear Hybrid Map on Generalized Space

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Abstract. In a recent paper Pathak *et al.* [20] established the coincidence and fixed point theorems for nonlinear hybrid contraction map under f-weak compatible continuous maps on metric spaces. In this paper we prove coincidence and fixed point theorems for nonlinear hybrid contraction maps on generalized metric spaces for multi-valued and single maps. Proved results of this paper to be a substantial generalization of the corresponding theorem of the recent paper [20].

1. **Introduction**

There are many coincidence and fixed point theorems for nonlinear hybrid contraction maps of a closed and bounded subset $CB(X)$ for a complete metric space X. However, in many applications, the maps involved may refer to Hadzic [5], Jungck [8], Kaneko *et al.* [9–11], Kannan [12], Pathak *et al.* [17–22], so it is interest to determine sufficient conditions on nonlinear hybrid maps which sure the existence of a fixed point. Subsequently, a number of generalizations of the multi-valued contraction principle for non-linear hybrid contraction maps obtained may refer to Khan [13], Kubiak [14], Nadler [15], Naimpally *et al.* [16], Rhoades *et al.* [23], Sessa [24], Smithson [29]. In this paper we consider the hybrid of maps, viz., contractive conditions involving multi-valued and single maps on a generalized metric space satisfying very general contractive type conditions which include several general conditions studied by Hematulin and Singh [6], Pathak *et al.* [20, 21], Singh *et al.* [26]. The result of this paper is a substantial generalization of the corresponding Theorem 1.1 of the recent paper of Pathak, Khan and Cho [20].

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Theorem 1.1. *Let (X,d) be a complete metric space, let $f : X \to X$ and $P : X \to CB(X)$ be f-weak compatible continuous maps such that $P(X) \subset f(X)$ and*

$$H(Px,Py) \leq h[aL_1(x,y)+(1-a)N_1(x,y)] \quad \text{for all } x,y \text{ in } X,$$

where $0 \leq h < 1$, $0 \leq a < 1$,

$$L_1(x,y) = \max\left\{d(fx,fy), d(fx,Px), d(fy,Py), \frac{1}{2}[d(fx,Py)+d(fy,Px)]\right\}$$

and

$$N_1(x,y) = \Big[\max\{d^2(fx,fy), d(fx,Px)\cdot d(fy,Py), d(fx,Py)\cdot d(fy,Px), \frac{1}{2}[d(fx,Px)\cdot d(fy,Px)], \frac{1}{2}[d(fx,Py)\cdot d(fy,Py)]\}\Big]^{\frac{1}{2}}.$$

Then there exists a point $z \in X$ such that $fz \in Pz$, i.e. the point z is a coincidence point of f and P.

2. Preliminaries

In a sequel, we use the following notations and definitions.

Definition 2.1 (Czerwik [1–4])**.** Let X be (nonempty) a set and $s \geq 1$ a given real number. A function $d : X \times X \to R^+$ (nonnegative real) is called a b-metric provided that for all $x,y,z \in X$,

- **(bm-1)** $d(x,y) = 0$, iff $x = y$,
- **(bm-2)** $d(x,y) = d(y,x)$,
- **(bm-3)** $d(x,z) \leq s[d(x,y)+d(y,z)]$.

The pair (X,d) is called a b-metric space.

We remark that a metric space is evidently a b-metric space. However, Czerwik [1,2] has shown that a b-metric on X need not be a metric on X (see also [3,27]). The following example shows that b-metric on X need not be a metric on X.

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Example 2.1. Let $X = \{x_1,x_2,x_3\}$ and $d : X \times X \to R^+$ such that

$$d(x_1,x_2)=x \geq 3,\ d(x_1,x_3)=d(x_2,x_3)=1,\ d(x_n,x_n)=0,\ d(x_n,x_k)=d(x_k,x_n).$$

Then

$$d(x_n,x_k) \leq \frac{x}{3}[d(x_n,x_i)+d(x_i,x_k)], \quad n,k,i,=1,2,3.$$

Then (X,d) is a b-metric space.

Definition 2.2 (Czerwik [2])**.** Let (X,d) be a b-metric space. The Hausdorff b-metric H on $CL(X)$, the collection of all nonempty closed subsets of (X,d) is defined as follows:

$$H(A,B) := \left\{\max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \text{if the maximum exists, otherwise } \infty\right\}.$$

In all that follows Y is an arbitrary nonempty set and (X,d) a b-metric space unless otherwise specified.

For the following definition in a metric space, one may refer to Itoh and Takahashi [7], and Singh and Mishra [28].

Definition 2.3. Let Y be a nonempty set, $f : Y \to Y$ and $P : Y \to 2^Y$, the collection of all nonempty subsets of Y. Then the hybrid pair (P,f) is (IT)-commuting at $z \in Y$ if $fPz \subseteq Pfz$ for each $z \in Y$.
Let (X,d) be a metric space and let $f : Y \to Y$ and $P,Q : Y \to CL(X)$ be single-valued and multivalued maps respectively.

We cite the following lemmas from Czerwik [1,2], and Singh *et al.* [27].

Lemma 2.1. *For any $A,B,C \in CL(X)$,*

(i) $d(x,B) \le d(x,y)$ *for any* $y \in B$,
(ii) $d(A,B) \le H(A,B)$,
(iii) $d(x,B) \le H(A,B)$, $x \in A$
(iv) $H(A,C) \le s[H(A,B) + H(B,C)]$,
(v) $d(x,A) \le sd(x,y) + sd(y,A)$, $x,y \in X$.

Lemma 2.2. *Let $A,B \in CL(X)$ and $k > 1$. Then for each $a \in A$, there exists a point $b \in B$ such that $d(a,b) \le kH(A,B)$.*

3. Coincidence Point Theorems

We start with following theorem.

Theorem 3.1. *Let (X,d) be a complete b-metric space, let $f : Y \to Y$ and $P,Q : Y \to CL(X)$ be maps such that $P(Y) \cup Q(Y) \subset f(Y)$*

$$H(Px,Qy) \le h[aL(x,y) + (1-a)N(x,y)] \tag{3.1}$$

for all x,y in X, where $0 \le h,a < 1$,

$$L(x,y) = \max\left\{d(fx,fy), d(fx,Px), d(fy,Qy), \frac{1}{2}[d(fx,Qy) + d(fy,Px)]\right\} \tag{3.2}$$

and

$$N(x,y)=\Bigg[\max\Bigg\{d^2(fx,fy),d(fx,Px)\cdot d(fy,Qy),d(fx,Qy)\cdot d(fy,Px),\\ \frac{1}{2}[d(fx,Px)\cdot d(fy,Px)]\Bigg\}\Bigg]^{\frac{1}{2}}. \tag{3.3}$$

If $s\sqrt{h}<1$, one of $P(Y)$, $Q(Y)$ or $f(Y)$ is a complete subspace of X, then $fx\in Px\cap Qx$ has a solution. Indeed, for any $x_0\in Y$, there exists a sequence $\{x_n\}$ in Y such that the sequence $\{fx_n\}$ converges to fz for some $z\in Y$, and $fz\in Pz\cap Qz$.

Proof. If $s=1$ then the conclusion follows from metric space setting, so we need to take $s>1$. Pick $x_0\in Y$. We construct sequences $\{x_n\}$ in Y and $\{fx_n\}$ in X in the following manner. Since $P(Y)\subseteq f(Y)$, we can find a point $x_1\in Y$ such that $fx_1\in Px_0$. Noting that $Q(Y)$ is also a subspace of $f(Y)$, for a suitable point $x_2\in Y$, we can choose a point $fx_2\in Qx_1$ such that

$$d(fx_1,fx_2)\leq kH(Px_0,Qx_1),\quad \text{where } k=h^{-1/2}.$$

In general, we can choose a sequence $\{x_n\}$ in Y such that $fx_{2n+1}\in Px_{2n}$, $fx_{2n+2}\in Qx_{2n+1}$, $fx_{2n+3}\in Px_{2n+2}$ and

$$\begin{aligned} d(fx_{2n+1},fx_{2n+2}) &\leq kH(Px_{2n},Qx_{2n+1}),\\ &\leq kh[aL(x_{2n},x_{2n+1})+(1-a)N(x_{2n},x_{2n+1})] \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} L(x_{2n},x_{2n+1}) &\leq \max\Bigg\{d(fx_{2n},fx_{2n+1}),d(fx_{2n},fx_{2n+1}),d(fx_{2n+1},fx_{2n+2}),\\ &\qquad \frac{1}{2}[d(fx_{2n},fx_{2n+2})+d(fx_{2n+1},fx_{2n+1})]\Bigg\}\\ &\leq \max\Bigg\{d(fx_{2n},fx_{2n+1}),d(fx_{2n+1},fx_{2n+2}),\\ &\qquad \frac{1}{2}s[d(fx_{2n},fx_{2n+1})+d(fx_{2n+1},fx_{2n+2})]\Bigg\} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &N(x_{2n},x_{2n+1})\\ &\leq \big[\max\{d^2(fx_{2n},fx_{2n+1}),d(fx_{2n},fx_{2n+1})\cdot d(fx_{2n+1},fx_{2n+2}),0,0\}\big]^{1/2}. \end{aligned} \tag{3.6}$$

Now by equation (3.4), (3.5) and (3.6), we get

$$d(fx_{2n+1},fx_{2n+2})\leq kh[asd(fx_{2n},fx_{2n+1})+(1-a)0].$$

Suppose that $d(fx_{2n+1},fx_{2n+2})>khasd(fx_{2n},fx_{2n+1})$ for some $n\in N$. Then we obtain $d(fx_{2n+1},fx_{2n+2})<d(fx_{2n},fx_{2n+1})$, which is a contradiction, and so $d(fx_{2n+1},fx_{2n+2})\leq as\sqrt{h}d(fx_{2n},fx_{2n+1})$.

Similarly $d(fx_{2n+2}, fx_{2n+3}) \leq as\sqrt{h}d(fx_{2n+1}, fx_{2n+2})$. Therefore in general $d(fx_{n+1}, fx_{n+2}) \leq as\sqrt{h}d(fx_n, fx_{n+1})$, for all $n \in N$. Since $a < 1$, $s\sqrt{h} < 1$ and X is complete, it follows from (3.4) that $\{fx_n\}$ is a Cauchy sequence. If we assume that $f(Y)$ is a complete subspace of X, then the sequence $\{x_n\}$ and its subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have a limit in $f(Y)$. Call it u. Then there exists a point $z \in Y$ such that $fz = u$. By (3.1) and Lemma 2.2, we have

$$\begin{aligned} d(fz, fx_{2n+2}) &\leq kH(Pz, Qx_{2n+1}) \\ &= kh[aL(z, x_{2n+1}) + (1-a)N(z, x_{2n+1}) \end{aligned} \tag{3.7}$$

where

$$L(z, x_{2n+1}) \leq \max\left\{d(fz, fx_{2n+1}), d(fz, Pz), d(fx_{2n+1}, fx_{2n+2}), \frac{1}{2}[d(fz, fx_{2n+2}) + d(fx_{2n+1}, Pz)]\right\}$$

and

$$N(z, x_{2n+1}) \leq \left[\max\left\{d^2(fz, fx_{2n+1}), d(fz, Pz)\cdot d(fx_{2n+1}, fx_{2n+2}), d(fz, fx_{2n+2})\cdot d(fx_{2n+1}, Pz), \frac{1}{2}[d(fz, Pz)\cdot d(fx_{2n+1}, Pz)]\right\}\right]^{1/2}.$$

Making $n \to \infty$, we have

$$\begin{aligned} L(z, x_{2n+1}) &\leq \max\left\{d(fz, fz), d(fz, Pz), d(fz, fz), \frac{1}{2}[d(fz, fz) + 0]\right\} \\ &\leq \max\{0, d(fz, Pz), 0, 0\} \\ &= d(fz, Pz) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} N(z, x_{2n+1}) &\leq \left[\max\left\{d^2(fz, fz), d(fz, Pz)\cdot d(fz, fz), d(fz, fz)\cdot d(fz, Pz), \frac{1}{2}[d(fz, fz)\cdot d(fz, fz)]\right\}\right]^{1/2} \\ &\leq [\max\{0, 0, 0, 0, 0\}]^{1/2} \end{aligned} \tag{3.9}$$

respectively. Thus we have from (3.7), (3.8), (3.9)

$$\begin{aligned} d(fz, Pz) &\leq khad(fz, Pz) \\ &= a\sqrt{h}d(fz, Pz). \end{aligned}$$

Which implies $d(fz, Pz) = 0$, because $a\sqrt{h} \leq 1$ therefore $fz \in Pz$, since Pz is closed. Similarly $fz \in Qz$, Thus $fz \in Pz \cap Qz$. This completes the proof. □

Remark 3.1. Take $P = Q$ the identity maps, in Theorem 3.1, we obtain generalizations of several coincidence results existing in the literature (see, for instance [6], [25], [26]).

4. Fixed Point Theorems

We apply coincidence theorem of the previous section to study fixed point theorem.

Theorem 4.1. *Let all the hypotheses of Theorem* 3.1 *be satisfied with* $Y = X$*. If* f *is* (IT)*-commuting with each of* P *and* Q *at their common coincidence point* z*, and if* $u = fz$ *is fixed point of* f*, then* f*,* P *and* Q *have a common fixed point, i.e.,*

$$u = fu \in Pu \cap Qu\,.$$

Proof. It comes from Theorem 3.1 that there exist $z, u \in X$ such that $u = fz \in Pz$ and $u = fz \in Qz$. Since $u = fu$, the (IT)-commutativity of f and P implies that $u = fu = ffz \in fPz \subseteq Pfz = Pu$. Similarly $u = fu \in Qu$. So $u = fu \in Pu \cap Qu$. This completes the proof. □

Remark 4.1. Let all the hypotheses of Theorem 4.1 be satisfied with $Y = X$. If f is (IT)-commuting with each of $P = Q$ at their common coincidence point z, and if $u = fz$ is fixed point of f, then $f, P = Q$ have a common fixed point, i.e.,

$$u = fu \in Pu\,.$$

Remark 4.2. If we take $k = h^{-1/2}$ in proof Theorem 3.1 at $k > 1$ then it to be $skh < 1$. So $k > 1$ in Theorems 3.1 and 4.1. Then we can take $s\sqrt{h} < 1$ at $skh < 1$. If we change condition $N(x, y)$ with condition $N_1(x, y)$ then condition $s\sqrt{h} < 1$ will change with condition $sh^{2/3} < 1$ and make some corrections. So we can take $skh < 1$ at $s\sqrt{h} < 1$, where $k > 1$.

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